P Vance

Mathematical Induction and Conic Sections

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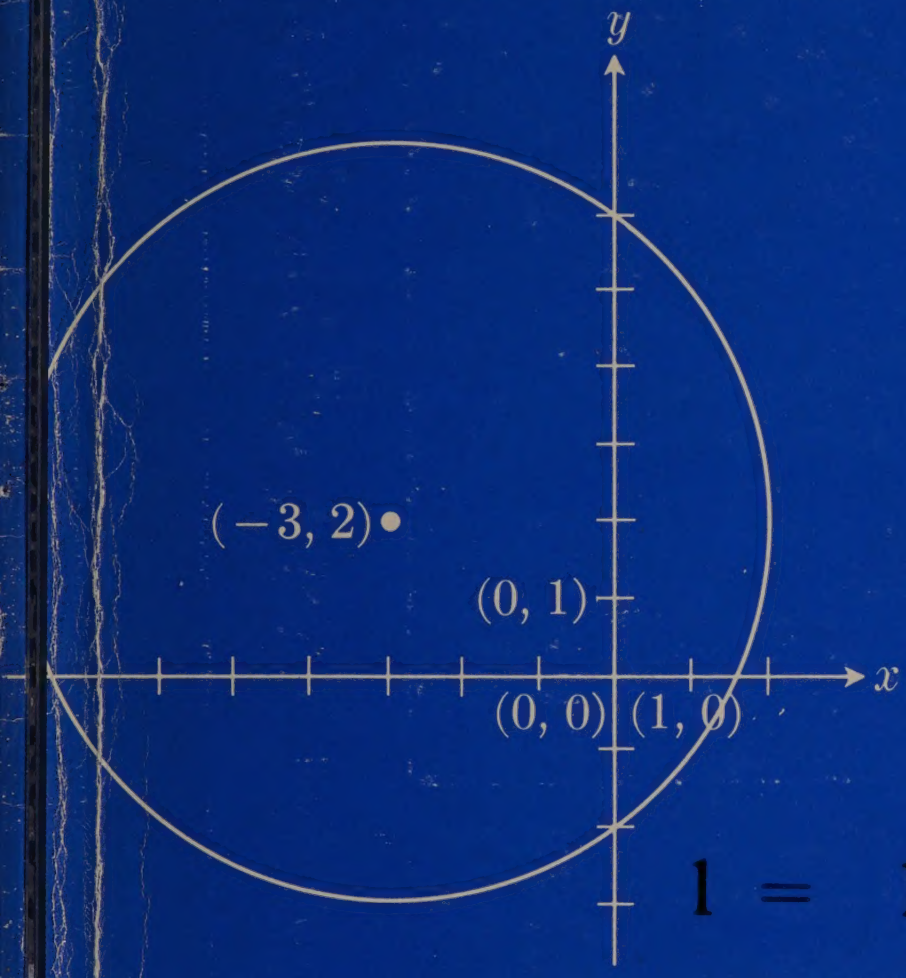

$$1 = 1 = 1^2,$$

$$1 + 3 = 4 = 2^2,$$

$$1 + 3 + 5 = 9 = 3^2,$$

$$1 + 3 + 5 + 7 = 16 = 4^2.$$

Mathematical Induction and Conic Sections are, respectively, chapters 11 and 20 in the 1968 edition of *An Introduction to Modern Mathematics* by Elbridge P. Vance. These chapters, plus the answers to chapter 20, have been brought together in this book.

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Answers to Problems of Chapter Twenty

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MATHEMATICAL INDUCTION

AND

CONIC SECTIONS

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11

MATHEMATICAL INDUCTION

MATHEMATICAL INDUCTION

AN ALTERNATIVE PROOF OF THE BINOMIAL THEOREM

REVIEW PROBLEMS

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11–1 MATHEMATICAL INDUCTION

One of the most important methods of proof in mathematics is that of mathematical induction. The name is unfortunate, for we are likely to associate the word induction with inductive reasoning, by which a conclusion is drawn from a large number of special cases. No certain conclusion can be reached by the inductive logical process. Actually, mathematical induction is deductive in nature, for it leads to a firm conclusion. Since it will be a convenient method of proof for several important results that appear in our work, we shall consider the method in some detail. It is usually employed in proving the validity of a statement involving all positive integral values of n.

Let us illustrate this method of proof. Assume that we have an ordinary ladder with an indefinite number of steps and that we wish to prove that we can climb to any designated step. We can do this if we know two things:

a) we can climb to the first step

b) if we are on any step, we can climb to the next higher step.

Because of (a) we know that we can get on the first step. With this knowledge and the fact stated in (b), we know we can get on the second step. Again, with this knowledge and the general fact stated in (b), we know we can get on the third step, etc.

One of the methods of finding a square root on an ordinary computing or adding machine is based on the fact that the sum of n odd integers is equal to n^2. Specifically,

$$\begin{aligned} 1 &= 1 = 1^2, \\ 1 + 3 &= 4 = 2^2, \\ 1 + 3 + 5 &= 9 = 3^2, \\ 1 + 3 + 5 + 7 &= 16 = 4^2. \end{aligned}$$

This property is true for all integers, and is not difficult to prove.

Example 1. Prove by mathematical induction that for all positive integral values,

$$1 + 3 + 5 + \cdots + (2n - 1) \equiv n^2.$$

Solution. Part (a): *Verification for a specific value.* We have done this for $n = 1$, 2, 3, and 4, although such verification is necessary only for one value.

Part (b): *Induction property.* If the statement is true for $n = k$, where k denotes any value of n, then we wish to prove that it is true for $n = k + 1$. Thus we assume we know this fact for $n = k$, namely,

$$1 + 3 + 5 + \cdots + (2k - 1) \equiv k^2.$$

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Adding $2k + 1$ to both members, we have

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) \equiv k^2 + (2k + 1)$$
$$\equiv (k + 1)^2.$$

This is precisely the equation $1 + 3 + 5 + \cdots + (2n - 1) \equiv n^2$ stated for $n = k + 1$. Thus the proof of Part (b) is complete.

Part (c): *Conclusion.* We know that the equation is true for $n = 1$, 2, 3, and 4. Since it is true for $n = 4$, and since we have proved the statement in Part (b), we know the original equation is true for $n = 4 + 1 = 5$; since it is true for $n = 5$, it is also true for $n = 6$. We may reason in this manner for all positive integral values of n.

We observe that a proof by mathematical induction consists of three parts.

Part (a): *Verification* of the validity of the statement or theorem for the smallest integral value of n for which the theorem is to hold. (This is analogous to being able to climb to the first step of the ladder.)

Part (b): *Proof of the inductive property.* If the statement or theorem is valid for $n = k$, where k denotes any value of n, then it is valid for $n = k + 1$. (This is analogous to being able to climb from any step of the ladder to the next higher step.)

Part (c): *Conclusion.* The statement or theorem is valid for all integral values of n equal to or greater than that for which it was verified in Part (a).

We know that $x - y$ is a factor of $x - y$, $x^2 - y^2$, and $x^3 - y^3$. Is it a factor of $x^n - y^n$ for any positive integer n?

Example 2. Prove by mathematical induction that $x^n - y^n$ is divisible by $x - y$ for all positive integral values of n.

Solution. Part (a): When $n = 1$, $x^n - y^n$ becomes $x - y$, which is clearly divisible by $x - y$. When $n = 2$, $x^2 - y^2 \equiv (x + y)(x - y)$, which again is divisible by $x - y$.

Part (b): We assume that $x^k - y^k$ is divisible by $x - y$. With this assumption, we must show that $x^{k+1} - y^{k+1}$ is also divisible by $x - y$. If we add and subtract xy^k to $x^{k+1} - y^{k+1}$, we have

$$x^{k+1} - y^{k+1} \equiv x^{k+1} - xy^k + xy^k - y^{k+1}$$
$$\equiv x(x^k - y^k) + y^k(x - y).$$

Since each term of the right member of this identity is divisible by $x - y$, the right and therefore the left member is divisible by $x - y$. This completes Part (b).

Part (c): We know that the theorem is true for $n = 1$ and 2. Since it is true for $n = 2$, by Part (b) it is true for $n = 2 + 1 = 3$. Similar reasoning applies for all positive integral values of n.

The fact that both Parts (a) and (b) are necessary for complete proof must be stressed. Consider the following examples.

Example 3. In Article 1–4 we considered prime numbers. Does the expression $n^2 - n + 11$ represent a prime number for all positive values of n? For $n = 1$, 2, 3, 4, and 5, we have the prime numbers 11, 13, 17, 23, and 31. Certainly Part (a) is satisfied for our statement. To show that the statement is not valid, however, let $n = 11$, and we have $11^2 - 11 + 11 = 11^2$, which is not a prime number.

Example 4. For even integers, let us assume the nonvalid statement that

$$2 + 4 + 6 + \cdots + 2n = n^2 + n + 1$$

is true when $n = k$. We can then prove the statement's validity, given this basic assumption, for $n = k + 1$. We have

$$2 + 4 + 6 + \cdots + 2k = k^2 + k + 1.$$

Adding $2k + 2$ to both members of the equality, we obtain

$$\begin{aligned} 2 + 4 + 6 + \cdots + 2k + 2k + 2 &= k^2 + k + 1 + 2k + 2 \\ &= k^2 + 2k + 1 + k + 1 + 1 \\ &= (k + 1)^2 + (k + 1) + 1. \end{aligned}$$

Thus, if the statement is valid for $n = k$, it is also valid for $n = k + 1$, and Part (b) has been proved. It is clear, however, that Part (a) and thus the statement itself is not valid, for consider the case where $n = 1$, 2, or 3. We would have $2 = 3$, $6 = 7$, or $12 = 13$, respectively.

We should emphasize that many proofs by mathematical induction are possible without adding the same quantity to both members of an equation known to be true. (This was the method in Example 1.)

PROBLEMS

By the method of mathematical induction, prove that the following are valid for all positive integral values of n.

1. $1 + 2 + 3 + \cdots + n = \dfrac{n(n + 1)}{2}$

2. $2 + 4 + 6 + \cdots + 2n = n(n + 1)$

3. $4 + 8 + 12 + \cdots + 4n = 2n(n + 1)$

4. $1 + 4 + 7 + \cdots + (3n - 2) = \dfrac{n(3n - 1)}{2}$

5. $1 + 3 + 6 + \cdots + \dfrac{n(n+1)}{2} = \dfrac{n(n+1)(n+2)}{6}$

6. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

7. $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$

8. $\dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \cdots + \dfrac{1}{2^n} = 1 - \dfrac{1}{2^n}$

9. $2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$

10. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$

11. $1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2 - 1)$

12. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$

13. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$

14. $1 \cdot 2 + 3 \cdot 4 + 5 \cdot 6 + \cdots + (2n-1)(2n) = \dfrac{n(n+1)(4n-1)}{3}$

15. $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \cdots + n(n+1)(n+2) = \dfrac{n(n+1)(n+2)(n+3)}{4}$

16. $t_1 + t_1 r + t_1 r^2 + \cdots + t_1 r^{n-1} = \dfrac{t_1 - t_1 r^n}{1 - r}$

17. $t_1 + (t_1 + d) + (t_1 + 2d) + \cdots + t_1 + (n-1)d = \dfrac{n[2t_1 + (n-1)d]}{2}$

18. $x^{2n} - y^{2n}$ is divisible by $x - y$

19. $x^{2n-1} + y^{2n-1}$ is divisible by $x + y$

20. $n^3 + 2n$ is divisible by 3

21. If $a > -1$, $(a+1)^n \geq 1 + na$

22. $\sin(\theta + n\pi) = (-1)^n \sin\theta$

23. $\cos(\theta + n\pi) = (-1)^n \cos\theta$

24. If n is a positive odd integer, $n(n^2 - 1)$ is divisible by 24.

25. $10^n + 3 \cdot 4^{n+2} + 5$ is divisible by 9

26. $2n \leq 2^n$

27. $2^{n-1} \leq n!$

28. Prove by mathematical induction the generalization of Eq. (4–6), namely, that for an algebraic sum of any fixed number of terms

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|.$$

Prove the following by the method of mathematical induction.

29. $\sin\theta + \sin 2\theta + \cdots + \sin n\theta \equiv \dfrac{\sin \frac{1}{2}(n+1)\theta \sin \frac{1}{2}n\theta}{\sin \theta/2}$

30. $\cos\theta + \cos 2\theta + \cdots + \cos n\theta \equiv \dfrac{\cos \frac{1}{2}(n+1)\theta \sin \frac{1}{2}n\theta}{\sin \theta/2}$

31. $\sin\theta + \sin 3\theta + \cdots + \sin(2n-1)\theta \equiv \dfrac{\sin^2 n\theta}{\sin\theta}$

32. $\cos\theta + \cos 3\theta + \cdots + \cos(2n-1)\theta \equiv \dfrac{\sin 2n\theta}{2\sin\theta}$

33. Recursion formulas are helpful in many fields of mathematics. One such formula makes it possible to find the value of $\sin nt$ for any integer n if $\sin t$ and $\cos t$ are known. If $u_0 = 0$, $u_1 = 1$, and $u_n = (2\cos t)u_{n-1} - u_{n-2}$ for $n = 2, 3, \ldots$, prove that $u_n = \sin nt/\sin t$ for $n = 0, 1, 2, \ldots$.

34. For the same values of u_n given in Problem 33, prove $(\cos t)u_n - u_{n-1} = \cos nt$ for $n = 0, 1, 2, \ldots$. This formula enables us to find $\cos nt$ if $\sin t$ and $\cos t$ are known. Such formulas are frequently used in work involving high-speed computing machines, because of the repetitive nature of the process.

35. In order to solve this problem, several completely different but important ideas must be used. The method of solution illustrates that many concepts are frequently needed to obtain one result. Let us consider the Fibonacci numbers defined by the recursion formula $u_0 = 0$, $u_1 = 1$, and $u_n = u_{n-1} + u_{n-2}$ for $n = 2, 3, 4, \ldots$. By direct computation, we find the sequence of u_n's to be: $0, 1, 1, 2, 3, 5, 8, 13, \ldots$.

a) Form the quotients: $u_1/u_2, u_3/u_4, u_5/u_6, u_7/u_8$.

b) Also form the quotients: $u_2/u_3, u_4/u_5, u_6/u_7, u_8/u_9$.

c) Verify that the quotients in (a) decrease in size, that the quotients in (b) increase, and that both sets of quotients appear to "approach some value" between 0.625 and 0.615.

d) Prove by induction: $u_1/u_2 > u_3/u_4 > \cdots > u_{2n-1}/u_{2n} > u_{2n+1}/u_{2n+2} \ldots$; we call $\{u_{2n-1}/u_{2n}\}$ a *decreasing* sequence.

e) Prove by induction $u_2/u_3 < u_4/u_5 < \cdots < u_{2n}/u_{2n+1} < u_{2n+2}/u_{2n+3} \ldots$; we call $\{u_{2n}/u_{2n+1}\}$ an *increasing* sequence.

f) Prove $u_{2n-1}/u_{2n} > u_{2m}/u_{2m+1}$ where m and n are any positive integers. Thus, the set $\{u_{2n-1}/u_{2n}\}$ is bounded below so that by the Completeness Property (Article 4-3) there exists a greatest lower bound G of this set. Similarly, the set $\{u_{2n}/u_{2n+1}\}$ is bounded above so that there exists a least upper bound L of this set.

g) Show that for n sufficiently large, the expression

$$\left|\frac{u_{2n-1}}{u_{2n}} - \frac{u_{2n}}{u_{2n+1}}\right| = \left|\frac{u_{2n+1}u_{2n-1} - u_{2n}^2}{u_{2n+1}u_{2n}}\right|$$

can be made as small as we please. *Hint:* Show (1) as n increases, the denominator increases, and (2) $u_{2n+1}u_{2n-1} - u_{2n}^2 = 1$ for all positive integers n. (Use induction.) As a result of (f) and (g) we know that G, the greatest lower bound

of $\{u_{2n-1}/u_{2n}\}$ is equal to L, the least upper bound of $\{u_{2n}/u_{2n+1}\}$. Let us denote this value by x, so that $G = L = x$. In addition, for n sufficiently large, there exist elements of both these sets arbitrarily close to x.

h) If we divide the original defining recursion formula

$$u_{2n+1} = u_{2n} + u_{2n-1} \quad \text{by} \quad u_{2n},$$

we have

$$\frac{u_{2n+1}}{u_{2n}} = 1 + \frac{u_{2n-1}}{u_{2n}} \quad \text{or} \quad \frac{1}{\dfrac{u_{2n}}{u_{2n+1}}} = 1 + \frac{u_{2n-1}}{u_{2n}}.$$

For n sufficiently large, we may approximate this equation by substituting x for u_{2n}/u_{2n+1} and u_{2n-1}/u_{2n}, so that $1/x = 1 + x$, or $x^2 + x - 1 = 0$. Solve this equation for the positive root x (why are we interested in the positive root?) and thus show that $x = G = L = (-1 + \sqrt{5})/2 = .618$ (approximately). [Recall Eq. (6–19) and Problem 24, Article 8–14.]

11–2 AN ALTERNATIVE PROOF OF THE BINOMIAL THEOREM

It is quite possible to prove the binomial theorem by mathematical induction. The proof is given below and involves a certain facility with algebra. We give it to emphasize the importance of proofs by mathematical induction and to illustrate another important proof by induction, which does not involve the process of adding the same expression to both sides of an equation known to be true.

Theorem 11–1. For any positive integer n,

$$(a+b)^n \equiv a^n + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots + b^n. \tag{11-1}$$

Proof. We recall Eq. (10–11), the general expression for the rth term:

$$\frac{n(n-1)(n-2)\cdots(n-r+2)}{(r-1)!}a^{n-r+1}b^{r-1}. \tag{11-2}$$

We are now prepared to prove that Eq. (11–1) holds for all positive integral values of n. Since we have verified it for $n = 1, 2, 3$, and 4 in Article 10–4, we need only to assume that it is valid for $n = k$, that is,

$$(a+b)^k \equiv a^k + ka^{k-1}b + \frac{k(k-1)}{2!}a^{k-2}b^2 + \cdots + \frac{k(k-1)(k-2)\cdots(k-r+2)}{(r-1)!}a^{k-r+1}b^{r-1} + \cdots + b^k, \tag{11-3}$$

and prove from this that it is valid for $n = k + 1$, or

$$(a+b)^{k+1} \equiv a^{k+1} + (k+1)a^k b + \frac{(k+1)k}{2!}a^{k-1}b^2 + \cdots + \frac{(k+1)(k)(k-1)\cdots(k-r+3)}{(r-1)!}a^{k-r+2}b^{r-1} + \cdots + b^{k+1}. \tag{11-4}$$

If we multiply each member of the identity (11–3) by $(a + b)$, we have

$$\begin{aligned}(a+b)^{k+1} \equiv a^{k+1} &+ ka^k b + \cdots \\ &+ \frac{k(k-1)\cdots(k-r+2)}{(r-1)!}a^{k-r+2}b^{r-1} + \cdots \\ &+ ab^k + a^k b + \cdots \\ &+ \frac{k(k-1)\cdots(k-r+3)}{(r-2)!}a^{k-r+2}b^{r-1} + \cdots \\ &+ kab^k + b^{k+1}.\end{aligned}$$

The terms through ab^k represent the right member of Eq. (11–3) multiplied by a, and the remaining terms represent the right member multiplied by b. To obtain the rth term in the result, we have written the rth term in the first half of the right member and the $(r - 1)$st term in the second, thus both contain $a^{k-r+2}b^{r-1}$. If we simplify the whole right member, the total coefficient of $a^{k-r+2}b^{r-1}$ will be

$$\begin{aligned}&\frac{k(k-1)\cdots(k-r+2)}{(r-1)!} + \frac{k(k-1)\cdots(k-r+3)}{(r-2)!} \\ &= \frac{k(k-1)\cdots(k-r+3)}{(r-1)!}(k-r+2) \\ &\quad + \frac{k(k-1)\cdots(k-r+3)}{(r-1)(r-2)!}(r-1) \\ &= \frac{k(k-1)\cdots(k-r+3)}{(r-1)!}[(k-r+2)+(r-1)] \\ &= \frac{(k+1)k(k-1)\cdots(k-r+3)}{(r-1)!}.^*\end{aligned}$$

Thus the total right side is exactly that of Eq. (11–4). The conclusion, Part (c), should be clear, and the proof is complete.

---

* It is interesting to note here that we are adding $C(k, r-1)$ and $C(k, r-2)$, and by Problem 20, Article 10–3 this is the expected result, $C(k+1, r-1)$.

REVIEW PROBLEMS

1. The expression $1 + x + x^2 + \cdots + x^n$ (where $x \neq 1$) is identical to

a) $\dfrac{1 - x^{n+1}}{1 - x}$ b) x^{n+1} c) $\dfrac{1}{1 - x}$ d) $(1 + x)^n$

2. In order to obtain an inverse of a 3 by 3 non-singular matrix using the method outlined in Article 9–3, how many equations in how many variables must be solved? Outline a method of finding the inverse of a given 3 by 3 matrix, but do not carry out the details.

3. Find the characteristic polynomial for the matrix

$$\begin{bmatrix} 1 & -2 & 2 & -3 \\ 0 & 5 & 0 & -2 \\ 2 & 4 & -1 & 2 \\ 3 & -4 & 1 & -2 \end{bmatrix}.$$

4. Find the characteristic polynomial for the matrix

$$\begin{bmatrix} 2 & -4 & 2 & 1 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

From this example can you draw any generalization?

5. Recall that $n! = 1 \cdot 2 \cdot 3 \cdots (n - 1) \cdot n$. Show that

a) $\dfrac{(n + 4)(n + 3)!}{(n + 4)!} = 1$ b) $\dfrac{(2n + 2)!}{(2n - 1)!2n} = 2(n + 1)(2n + 1)$

6. Write the first four terms in the expansion $(1 - 1/x)^{10}$.

7. Use an appropriate binomial expansion to approximate $(0.99)^{20}$ correct to two decimal places.

8. Prove that the following is valid for all positive integral values of n:

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots n(n + 2) = \frac{n(n + 1)(2n + 7)}{6}.$$

9. Prove by mathematical induction that $10^n/3 + \frac{5}{3} + 4^{n+2}$ is divisible by 3.

10. Prove that for all integers $n \geq 3$, $(n + 1)^n < n^{n+1}$.

Prove (Problems 11 and 12) by mathematical induction that for all positive integral $n > 1$,

11. $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \cdots + \dfrac{1}{n^2} < 2 - \dfrac{1}{n}$ **12.** $\dfrac{1}{1^3} + \dfrac{1}{2^3} + \cdots + \dfrac{1}{n^3} < 2 - \dfrac{1}{n^2}$

13. How many different positive integers less than 1000 can be formed by using 2 odd digits and 1 even digit if zero is excluded, although repetitions of the other digits are allowed?

14. How many different positive integers less than 1000 can be formed by using 2 odd digits and 1 even digit if zero is allowed, along with other repetitions?

15. Express the matrix equation

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \quad \begin{bmatrix} 2 & -1 & 0 & 4 \\ 0 & -2 & 1 & 3 \\ 1 & 2 & -1 & 0 \\ 2 & 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ -5 \\ -7 \end{bmatrix}$$

in terms of four linear equations in x,y,z, and w, and solve.

16. Solve the following system of equations for x,y,z, and w, and write in the form of a matrix equation $AX = B$:

$$\begin{aligned} x - 2y + 3z - w &= 2, \\ 2x + 3y + w &= 10, \\ x + 4y - z &= 6, \\ 3x - 4y + z - 2w &= -10. \end{aligned}$$

17. a) Show that $x = 2$ is a double zero of f, where $f(x) = x^3 - 5x^2 + 3x + 9$.
b) Show that $x = 3$ is a single zero of f'.

18. Show that if a is a double zero of a function f, then a is a single zero of f'.

19. We recall in Article 3–3 that the process of division gave us the result [Eq. (3 – 8)],

$$\frac{P(x)}{D(x)} \equiv Q(x) + \frac{R(x)}{D(x)}.$$

Show that if $P(x)$ is a polynomial in x, $D(x)$ is in the form $x - a$, then this equation may be written

$$\frac{P(x)}{x - a} \equiv Q(x) + \frac{R}{x - a},$$

where Q is degree one, less than that of P, and R is a constant. This, of course, may also be written $P(x) \equiv (x - a)Q(x) + R$.

20. If $Q(x)$, the polynomial in Problem 19, is again divided by $x - a$, show that, in a similar way $Q(x)$ may be expressed

$$Q(x) \equiv (x - a)\, S(x) + T, \qquad \text{where } T \text{ is constant,}$$

so that

$$P(x) = (x - a)^2\, S(x) + (x - a)\, T + R.$$

21. Prove that $P'(x) = 2(x - a)\, S(x) + (x - a)^2\, S'(x) + T$, and thus $P'(a) = T$.

20

CONIC SECTIONS

GENERAL DISCUSSION • THE CIRCLE

THE PARABOLA • THE ELLIPSE

THE HYPERBOLA • SUMMARY AND APPLICATIONS

TRANSLATION OF AXES • ROTATION OF AXES

REVIEW PROBLEMS

20–1 GENERAL DISCUSSION

In many of the preceding articles we have considered the graphs of various curves. In Article 15–3 we discussed general methods that apply to all curves. In this chapter we shall consider in some detail four important special curves which occur so frequently in science that they deserve special mention. Each will be described geometrically, and it will then be shown that each is a graph of some quadratic equation in x and y; that is, each may be represented as a special case of the general equation

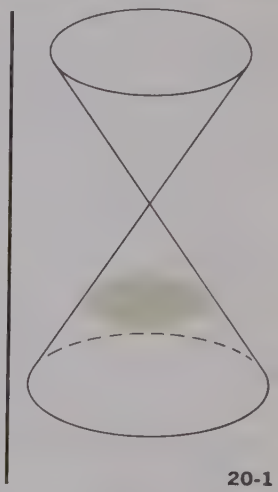
$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \qquad (20\text{–}1)$$

where A, B, and C are not all zero.

These four curves, the circle, parabola, ellipse, and hyperbola, are called *conic sections* (or simply *conics*) because they can be described as the curve of intersection of a plane and a right circular cone. In fact, this was the method first used in studying these curves and in establishing many of their properties.

20–2 THE CIRCLE

The circle is the simplest of the conic sections. Geometrically (Fig. 20–1), it is the intersection of a right circular cone and a plane perpendicular to the axis of the cone.



20-1

We recall from Article 4–6 the definition of a circle:

Definition 20–1. A *circle* is the locus of all points in the plane which are at a constant distance from a fixed point.

From this definition we obtained, by using Eq. (4–9), the general equation of a circle with center (h, k) and radius r [recall Eq. (4–9)],

$$(x - h)^2 + (y - k)^2 = r^2. \qquad (20\text{–}2)$$

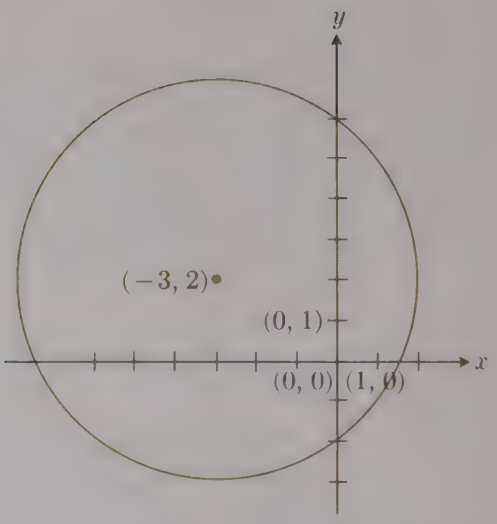
It would be helpful to recall the discussion and problems in Article 4–6.

Example 1. Identify and sketch the curve represented by the equation

$$x^2 + y^2 + 6x - 4y = 12.$$

Solution. We merely write this equation in the form of (20–2) by completing the square. (Recall Article 8–9.) We have

$$x^2 + y^2 + 6x - 4y = 12,$$
$$(x^2 + 6x + \quad) + (y^2 - 4y + \quad) = 12.$$



20-2

By adding 9 and 4 to both members of the equation, we get

$$(x^2 + 6x + 9) + (y^2 - 4y + 4) = 12 + 9 + 4,$$

or

$$(x + 3)^2 + (y - 2)^2 = 5^2.$$

The equation is now in the form of Eq. (20–2), with $h = -3$, $k = 2$, and $r = 5$, so that it represents a circle with center at $(-3, 2)$ and radius 5. Its graph is shown in Fig. 20–2.

This example indicates the use of two fundamental circle theorems and their methods of proof.

Theorem 20–1. An equation of any circle may be written

$$x^2 + y^2 + Dx + Ey + F = 0. \qquad (20–3)$$

Proof. If Eq. (20–2), which represents any circle, is expanded to

$$x^2 - 2hx + h^2 + y^2 - 2ky + k^2 = r^2$$

or

$$x^2 - 2hx + y^2 - 2ky = r^2 - h^2 - k^2,$$

by letting $-2h = D$, $-2k = E$, and $h^2 + k^2 - r^2 = F$, we have the desired result.

Theorem 20–2. Conversely, if Eq. (20–3) is satisfied by more than one point, it is an equation of a circle.

Proof. As in Example 1, we can complete the squares and put this equation in the form of Eq. (20–2).

$$x^2 + Dx + \frac{D^2}{4} + y^2 + Ey + \frac{E^2}{4} = \frac{D^2}{4} + \frac{E^2}{4} - F,$$

or

$$\left(x + \frac{D}{2}\right)^2 + \left(y + \frac{E}{2}\right)^2 = \frac{D^2 + E^2 - 4F}{4}.$$

If $D^2 + E^2 - 4F < 0$, the coordinates of no point will satisfy this equation. (Why?) If $D^2 + E^2 - 4F = 0$, only the point $(-D/2, -E/2)$ will satisfy the equation. (This is sometimes called a *point circle.*) If, however, $D^2 + E^2 - 4F > 0$, we have the equation of a circle whose center is $(-D/2, -E/2)$ and whose radius is $\frac{1}{2}\sqrt{D^2 + E^2 - 4F}$.

With this result, we may think of either Eq. (20–2) or (20–3) as representing any circle. The three independent constants h, k, and r, or D, E, and F, represent implicitly the fact that a circle is determined by three independent conditions, a familiar fact to students of plane geometry. This will be emphasized in the following examples as well as in the problems.

Example 2. Find an equation of the circle which passes through the three points $(7, -5)$, $(-1, 1)$, and $(0, -6)$.

Solution 1. Let us consider Eq. (20–3) as the solution to this example. Since the coordinates of any point which lies on a curve satisfy the equation of that curve, we must have the following three conditions satisfied:

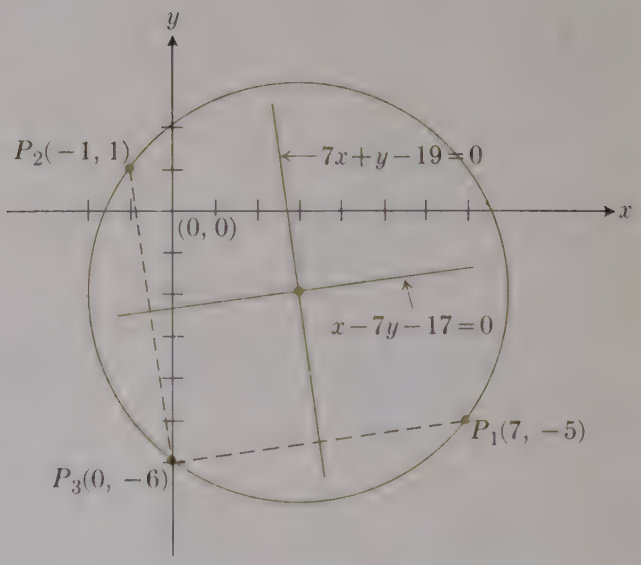
$$\begin{aligned} 49 + 25 + 7D - 5E + F &= 0, \\ 1 + 1 - D + E + F &= 0, \\ 36 - 6E + F &= 0. \end{aligned}$$

Solving this system of equations, we find $D = -6$, $E = 4$, and $F = -12$, so that an equation of the required circle is

$$x^2 - 6x + y^2 + 4y - 12 = 0.$$

Recall Problem 15, Article 8–7.

Solution 2. Observing Fig. 20–3, we know that the center of the required circle lies on the perpendicular bisector of P_1P_3 and also of P_2P_3. An equation of the perpendicular bisector of P_1P_3 is $7x + y - 19 = 0$, and of P_2P_3 is $x - 7y - 17 = 0$. Since the center (h, k) lies on both these lines, it lies at their point of intersection; hence $(h, k) = (3, -2)$. The radius r is the distance between $(3, -2)$ and any one of the three given points, that is, 5. Thus, using Eq. (20–2),



20-3

we have the result $(x - 3)^2 + (y + 2)^2 = 25$. It is easily seen that the two results are the same (why?).

Many problems of this general type are solved by finding two independent conditions that the coordinates of the center must satisfy, that is, two lines on which the center must lie. The solution of these two equations determines the center. Having found the center and the radius, we can use Eq. (20–2) to obtain the required equation of the circle.

PROBLEMS

1. Find equations of the circles whose centers and radii are given:

a) center (2, 3), radius 1;
b) center $(-6, 4)$, radius 6;
c) center $(-\frac{2}{3}, \frac{3}{2})$, radius $\frac{1}{4}$;
d) center $(2\sqrt{3}, -\sqrt{2})$, radius $\sqrt{5}$.

2. Determine whether the equation represents a circle, a point, or no locus. In the case of a circle, give the coordinates of the center and the radius and sketch.

a) $x^2 + y^2 + 4x - 6y + 9 = 0$,
b) $x^2 + y^2 - 6x + 8y + 24 = 0$,
c) $x^2 + y^2 + 8x - 14y + 65 = 0$,
d) $x^2 + y^2 - 3x + 7y - 14 = 0$,
e) $2x^2 + 2y^2 - 8x + 12y - 31 = 0$,
f) $36x^2 + 36y^2 - 48x - 36y + 16 = 0$.

In each of Problems 3 through 16 find equations of all circles satisfying the given conditions.

3. Center at the origin, passing through the point $(4, -5)$.

4. Center at $(-3, 2)$, passing through the point $(1, -4)$.

5. One of its diameters joins the points $(6, -8)$ and $(-2, 4)$.

6. One of its diameter joins the points $(3, 1)$ and $(-5, 7)$.

7. Center $(3, 2)$ and tangent to the line $x - 3y + 4 = 0$. *Hint:* Recall Eq. (7–17.)

8. Center $(-4, 1)$ and tangent to the line $2x + 3y - 7 = 0$.

9. Passing through the points $(5, 1)$, $(3, 3)$, and $(-1, -5)$.

10. Passing through the points $(5, 3)$, $(7, 1)$, and $(8, 2)$.

11. Passing through the points $(4, 9)$, $(5, 8)$, and $(-3, 2)$.

12. Passing through the points $(6, -6)$, $(-1, -5)$, and $(7, -5)$.

13. Passing through the point $(-2, 4)$ and tangent to the line $x - 2y = 2$ at the point $(6, 2)$.

14. Passing through the point $(3, -1)$ and tangent to the line $3x + 2y = 5$ at the point $(1, 1)$.

15. Passing through the points $(5, -3)$ and $(1, 3)$, with center on

$$x - 2y - 2 = 0.$$

16. Passing through the points $(2, 4)$ and $(1, -3)$, with center on $x + 2y = 0$.

17. Find the coordinates of the points of intersection of the circle $x^2 + y^2 - 4x + 6y - 12 = 0$ and the line $2x - y + 3 = 0$. (Recall Problem 11, Article 8–15.)

18. Find the coordinates of the points of intersection of the circle $x^2 + y^2 - 3x + y - 4 = 0$ and the line $2x + 3y + 5 = 0$.

19. If, as in Fig. 20–4,

$$x^2 + y^2 + D_1x + E_1y + F_1 = 0$$

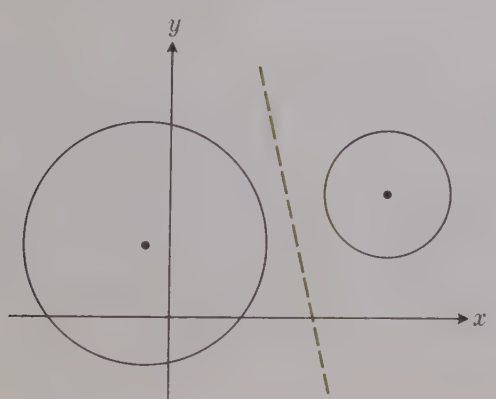
and

$$x^2 + y^2 + D_2x + E_2y + F_2 = 0$$

represent two nonconcentric circles, show that

a) $D_1 = D_2$ and $E_1 = E_2$ cannot both be possible;

b) if each member of the second equation is subtracted from the first equation, the resulting equation is a linear equation. The straight line represented by this equation is called the *radical axis of the two circles.* Find the radical axis of these two circles.



20-4

20. Prove that the radical axis of two circles is perpendicular to the line joining their centers. Choose the coordinate axes and the equations of the circles so as to make the work general and yet as simple as possible.

21. Find the radical axis of the circles $x^2 + y^2 - 4y = 0$ and $x^2 + y^2 - 6x + 8y + 24 = 0$. Does this radical axis pass through (6, 1)?

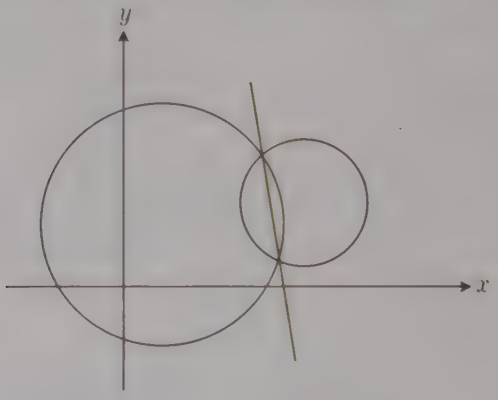
22. If the centers of three nonconcentric circles lie on a straight line, what can be said about the three radical axes on these circles, taken in pairs?

23. If their centers do not lie on a straight line, prove that the radical axes of three circles, taken in pairs, meet in a point. This point is called the *radical center*. Recall Problem 18, Article 9–4.

24. Find the radical center of the three circles

$$x^2 + y^2 - 4x = 0, \qquad x^2 + y^2 - 2y = 0, \qquad x^2 + y^2 + 8x + 2y - 32 = 0.$$

25. If the two circles in Problem 19 actually intersect, state why the corresponding radical axis must pass through the points of intersection of the two circles. This radical axis is then called the *line of their common chord*. The *common chord* is the line segment between the two points of intersection. See Fig. 20–5.



20-5

26. Find an equation of the common chord of the following circles and thus find the coordinates of their points of intersection. Then find the length of this common chord.

$$x^2 + y^2 - 2x - 4y = 20 \quad \text{and} \quad x^2 + y^2 + 2x = 52.$$

Find equations of all circles satisfying the given conditions in Problems 27 through 31. *Hint:* Find two independent conditions which the coordinates of the center must satisfy.

27. Passing through the points (2, −2) and (3, 5) and having 5 for its radius.

28. Passing through the points (−1, −3) and (3, 5), and tangent to the line $2x + y - 13 = 0$. *Hint:* The center of the required circle is equidistant from each of these points and the given line.

29. Passing through the point (6, 0) and tangent to the lines $4x - 3y - 12 = 0$ and $3x - 4y - 12 = 0$.

30. Passing through the point (4, 1), tangent to the line $x - y + 7 = 0$, and having a radius $\sqrt{32}$.

31. Passing through the point (1, 2), tangent to the line $x - 2y - 2 = 0$, and having a radius $\sqrt{5}$.

32. Find an equation of the tangent to each of the following circles at the point indicated on the circle (recall Article 13–4):

a) $x^2 + y^2 = 10$ at (1, 3),
b) $x^2 + y^2 - x + 7y + 6 = 0$ at (3, −3),
c) $x^2 + y^2 + 2x = 0$ at (0, 0),
d) $(x - 2)^2 + (y + 3)^2 = 25$ at (−1, 1).

33. Find equations of the tangents to the circles

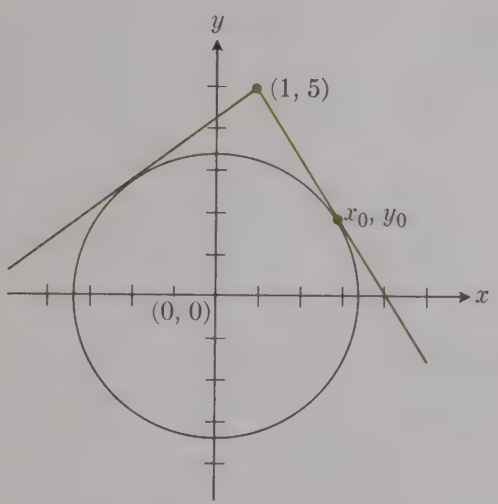
$$x^2 + y^2 - 3x + y = 0, \qquad x^2 + y^2 + 4x - 2y = 0$$

at one of their points of intersection and the angle of intersection of the circles at that point. Recall Problem 14, Article 13–4.

34. Prove that the circles

$$x^2 + y^2 - 4x + 2y - 4 = 0, \qquad x^2 + y^2 - 2x - 10y + 10 = 0$$

intersect at right angles.



20-6

35. Find equations of the tangents to the circle $x^2 + y^2 = 13$ through the point (1, 5), as shown in Fig. 20–6. *Hint:* Find the slope at any point (x_0, y_0), on the circle. Then find (x_0, y_0), since the line through (x_0, y_0) and (1, 5) is tangent to the circle and has this slope, and since (x_0, y_0) lies on the circle.

36. Find equations of the tangents to the circle $x^2 + y^2 + 4x - 4y - 2 = 0$ through the point (2, 4).

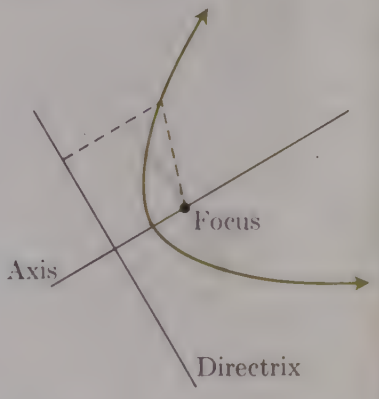
20–3 THE PARABOLA

As was mentioned in Article 20–1, each of the conic sections can be described geometrically as the intersection of a plane and a right circular cone. This intersection (see Fig. 20–7) is a parabola when the plane is parallel to a generator of the cone. However, as in the case of the circle, we shall define the parabola as a locus of points.

Definition 20–2. A *parabola* is the locus of all points in the plane which are equidistant from a fixed line and a fixed point not on the line.

The fixed line is called the *directrix*, while the fixed point is called the *focus* (Fig. 20–8). The line through the focus and perpendicular to the directrix is called the *axis of the parabola*, since the locus is clearly symmetric with respect to this line. The point on the axis midway between the directrix and the focus is the *vertex* of the curve.

20-7



20-8

To obtain an equation for this locus, and to make the work and also the result as simple as possible, let us choose the x-axis as the axis of the parabola and its vertex at the origin. Letting the distance between the vertex and focus be p, we may choose the focus F on the positive x-axis, with coordinates $(p, 0)$. Then the directrix will have $x = -p$ for its equation. By definition, any point $P(x, y)$ which lies on the curve of the parabola must be equidistant from the line $x = -p$ and the point $(p, 0)$ (Fig. 20–9). Therefore, recalling Eq. (8–18), we have

$$|x + p| = \sqrt{(x - p)^2 + (y - 0)^2}.$$

If we square both members and simplify, this equation becomes

$$y^2 = 4px. \tag{20–4}$$

Since the above algebraic steps may be reversed, any point whose coordinates satisfy Eq. (20–4) is a point satisfying the original definition. Thus Eq. (20–4) is an equation of a parabola with focus at $(p, 0)$ and directrix $x = -p$.

In Eq. (20–4), since $y^2 \geq 0$, $4px \geq 0$, so that if $p > 0$ the curve lies to the right of the y-axis. If $p < 0$, x must be negative, and the curve will lie to the left of the y-axis. The chord through the focus, perpendicular to the axis of the parabola, is called the *latus rectum* of the parabola. Since the ends of this chord are on the curve, and therefore are equidistant from the focus and directrix, the length of the latus rectum is $|4p|$. A rough sketch of any parabola in this form can be shown by using these two points together with the vertex, although other points should be located if a more accurate figure is required.

If, in deriving Eq. (20–4), the axis of the parabola had been chosen as the y-axis, the point $(0, p)$ as the focus, and thus $y = -p$ as the directrix, the corresponding equation would be

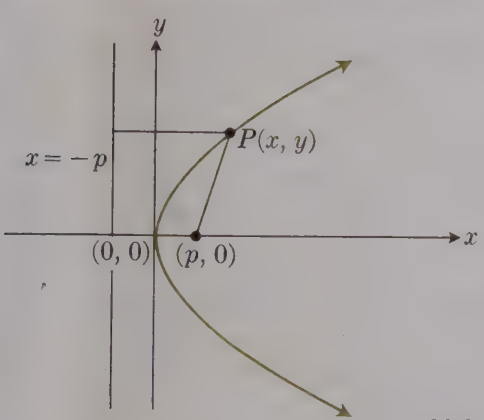
$$x^2 = 4py. \tag{20–5}$$

The parabola whose equation is given by Eq. (20–5) would lie above the x-axis if $p > 0$, but below if $p < 0$.

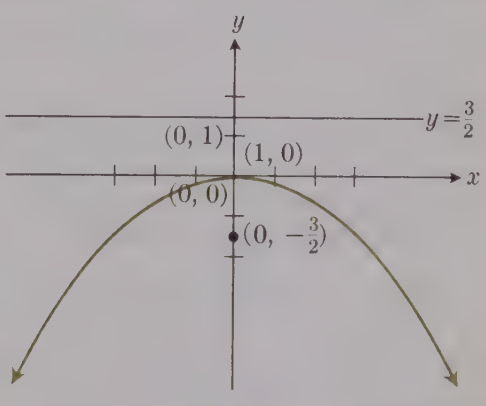
Example. Find the focus, directrix, and the length of the latus rectum of the parabola $x^2 = -6y$ and sketch.

Solution. Since this equation is in the form of Eq. (20–5) with $p = -\frac{3}{2}$, its axis is the y-axis, and it "opens down." Its focus has coordinates $(0, -\frac{3}{2})$; the directrix is $y = \frac{3}{2}$ and the length of the latus rectum is 6. The graph of the curve appears in Fig. 20–10.

It should be emphasized that the parabolas discussed in this article have one of the coordinate axes as their axis, and the origin as their vertex. In general, parabolas can have much more complicated equations, as will be seen in Articles 20–8 and 20–9.



20-9



20-10

PROBLEMS

Find the focus, directrix, length of the latus rectum, and sketch the graph of the parabolas in Problems 1 through 6.

1. $y^2 = 8x$ **2.** $x^2 = 9y$ **3.** $x^2 = -y$

4. $y^2 = -24x$ **5.** $2x^2 + 3y = 0$ **6.** $3y^2 + 7x = 0$

Find equations of the parabolas described in Problems 7 through 12.

7. Focus, $(2, 0)$; directrix, $x = -2$.

8. Focus, $(0, -3)$; directrix, $y = 3$.

9. Vertex, $(0, 0)$; focus, $(-\frac{3}{2}, 0)$.

10. Vertex $(0, 0)$; length of latus rectum 7; focus on the negative y-axis.

11. Vertex $(0, 0)$; focus on the x-axis; passing through $(2, 3)$.

12. Vertex $(0, 0)$; focus on the y-axis; passing through $(-3, 4)$.

Find equations of the parabolas described in Problems 13 through 16, using the original definition.

13. Focus $(1, 1)$; directrix $x + y + 2 = 0$.

14. Focus $(0, 0)$; directrix $x + y - 2 = 0$.

15. Focus $(3, 4)$; directrix $x = 2$.

16. Focus $(2, -3)$; directrix $y = -5$.

17. Using Definition 20–2, demonstrate that the equation $(y - k)^2 = 4p(x - h)$ represents the parabola whose vertex is at (h, k) and whose focus is p units to the right of (h, k). This equation will be discussed further in Article 20–8.

18. What would the conditions on the coefficients of Eq. (20–1) be if Eq. (20–1) represented the parabola in Problem 17?

19. The line segment between the focus F and any point P on the parabola is called a *focal radius*. Prove that for any point $P(x_0, y_0)$ on the parabola $y^2 = 4px$, the length of the focal radius is $|x_0 + p|$.

20. Find the locus of the center of a circle which is tangent to the line $y = -3$ and the circle $x^2 + y^2 = 4$.

21. Find the locus of the center of a circle which is tangent to the line $x = 2$ and the circle $x^2 + y^2 = 16$.

22. Find equations of the tangent and normal lines to the parabola $x^2 + 8y = 0$ at $(-4, -2)$, a point on the parabola.

23. Find equations of the tangents to the parabola $y^2 - 6x = 10$ through the external point $(5, 7)$. Draw the figure.

Problems 24 through 26 represent certain properties that are important in the use of the parabola.

24. Prove that the circle drawn with a focal radius of the parabola $y^2 = 4px$ as its diameter is tangent to the y-axis.

25. If PQ represents a chord of the parabola $y^2 = 4px$ passing through the focus, prove that the circle drawn with PQ as diameter is tangent to the directrix.

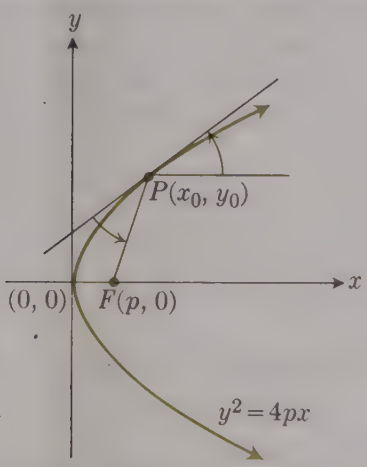
26. If any line is drawn tangent to the parabola $y^2 = 4px$ at $P(x_0, y_0)$, a point on the curve, prove that the angle between this tangent and the focal radius of the point $P(x_0, y_0)$ is equal to the angle between this tangent and the line through P, parallel to the x-axis. See Fig. 20–11.

27. If V is the vertex of a parabola, and $P(\neq V)$ any point on it, the line PV will meet the axis of the parabola at some point Q. If P' is the foot of the line from P perpendicular to the axis, show that $P'Q$ is equal in length to the latus rectum.

28. If the tangent to a parabola at any point P, on the parabola, meets the directrix at A and the latus rectum (extended) at B, show that A and B are equidistant from the focus.

29. If P is any point on a parabola whose focus is F, and the tangent and normal to the curve at P intersect its axis at A and B, respectively, show that $FA = FB$.

30. If the normal to a parabola at a point P meets the axis at A, show that P and A are equidistant from the focus. Prove that any angle inscribed in a semicircle is a right angle.



20-11

20-12

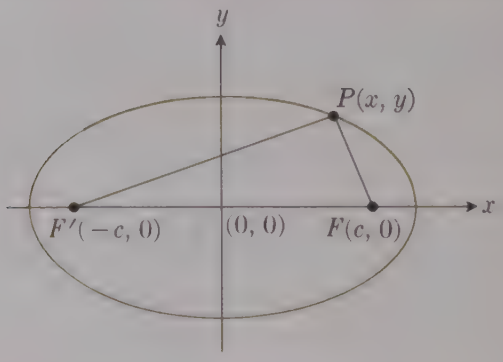
20–4 THE ELLIPSE

Geometrically, the ellipse can also be described as the intersection of a plane and a right circular cone. If the plane is inclined, but not parallel to a generator, so that it intersects only one nappe (section) of the cone, this intersection is an ellipse (see Fig. 20–12).

We again prefer to make use of a definition in terms of a locus of points:

Definition 20–3. An *ellipse* is the locus of all points in the plane such that the sum of its distances from two fixed points is a constant.

The two fixed points, usually denoted by F and F', are called the *foci* of the ellipse. The mid-point of the line segment FF' is called the *center* of the ellipse. If the length of this line segment is denoted by $2c$, and the sum of the distances (or constant mentioned in the definition) by $2a$, we have $2a > 2c$ or $a > c$. (Why?)



20-13

As in the case of the parabola, we wish to choose the coordinate axes so as to obtain an equation for the general ellipse in as simple a form as possible. If we take the center as the origin and the points $F(c, 0)$ and $F'(-c, 0)$ on the x-axis (Fig. 20–13), the definition requires that the coordinates of the point $P(x, y)$ satisfy the condition

$$|F'P| + |FP| = 2a,$$

or

$$\sqrt{(x + c)^2 + (y - 0)^2} + \sqrt{(x - c)^2 + (y - 0)^2} = 2a. \qquad (20\text{–}6)$$

If we transpose the first radical to the right side of the equation and square each member, we have

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + x^2 + 2cx + c^2 + y^2,$$

which becomes

$$4a\sqrt{(x + c)^2 + y^2} = 4a^2 + 4cx.$$

Dividing each member by 4, and again squaring, we eliminate the radical and have

$$a^2(x^2 + 2cx + c^2 + y^2) = a^4 + 2a^2cx + c^2x^2,$$

which reduces to

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2). \qquad (20\text{–}7)$$

Since $a > c$, $a^2 - c^2 > 0$. By denoting $a^2 - c^2$ by b^2 and dividing each member of Eq. (20–7) by a^2b^2, we get the result

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{20–8}$$

This calculation shows that any point $P(x, y)$ which lies on the ellipse has coordinates that satisfy Eq. (20–8). We also must show the converse, namely, that any point whose coordinates satisfy Eq. (20–8) lies on the ellipse. This may be done by reversing the steps of the above proof. Therefore Eq. (20–8) is the required equation and is called the *standard equation of the ellipse* with its center at the origin.

By considering the graph of this equation, as in Article 15–3, we find:

a) it is symmetric with respect to both axes, and thus to the origin;
b) its intercepts are $(\pm a, 0)$ and $(0, \pm b)$;
c) it lies in the rectangle where $|x| \leq a$ and $|y| \leq b$.

The points $V(a, 0)$ and $V'(-a, 0)$ are called the *vertices* of the ellipse, and the line segment VV' of length $2a$ is called the *major axis* of the ellipse. The line segment between the y-intercepts of length $2b$ is called its *minor axis*. The chord through either focus perpendicular to the major axis is called the *latus rectum;* it is of length $2b^2/a$ (why?). By using this length and the intercepts, one is able to sketch the graph of the curve with comparative ease.

As in the case of the parabola, if the foci had been located on the y-axis, namely $F(0, c)$ and $F'(0, -c)$, the major axis would lie along the y-axis, again of length $2a$, and the equation would become

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1, \tag{20–9}$$

with the vertices $V(0, a)$ and $V'(0, -a)$. In either case, Eq. (20–8) or (20–9), $b^2 = a^2 - c^2$, so that $a > b$.

Example 1. Sketch the graph of the ellipse whose equation is $9x^2 + 4y^2 = 36$, giving the coordinates of the foci, ends of the major and minor axes, and length of each latus rectum.

Solution. Dividing each member by 36, in order to put the equation in standard form, we have

$$\frac{x^2}{4} + \frac{y^2}{9} = 1,$$

so that $a = 3$ and $b = 2$, and the ellipse has its major axis along the y-axis (why?). Since $a^2 - c^2 = b^2$, $c = \sqrt{9 - 4} = \sqrt{5}$. The coordinates of the foci are $F(0, \sqrt{5})$ and $F'(0, -\sqrt{5})$, the vertices are $V(0, 3)$ and $V'(0, -3)$, and the ends

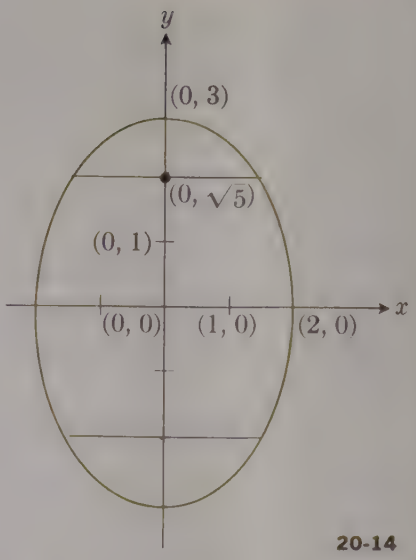
of the minor axis are (2, 0) and (−2, 0). The length of each latus rectum is

$$2b^2/a = 2 \cdot \tfrac{4}{3} = \tfrac{8}{3}.$$

The required graph appears in Fig. 20–14.

It should be clear that two ellipses would be similar (the same shape, but possibly of different sizes) if the ratio of b and a were constant. If, however, one of these remains fixed and the other changes, the actual shape of the ellipse would change. It is more useful to consider the ratio c/a. Since

$$\frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a} = \sqrt{1 - \frac{b^2}{a^2}},$$



20-14

if b/a remains constant, so does c/a, and conversely. This ratio c/a, denoted by e, is called the *eccentricity* of the ellipse. For any ellipse, the quantity e always has a value such that $0 \leq e < 1$. If we keep a fixed, that is, the location of the two vertices V and V', the shape of the ellipse ranges from a circle (where F and F' coincide, and $c = 0$, so $e = 0$) to a long, thin ellipse, hardly distinguishable from a straight line (where F and F' are close to V and V', or c is almost equal a). The limiting case where $c = a$ and $e = 1$ would indeed be the straight-line segment VV'. One of the important uses of the eccentricity appears in Problem 24.

Example 2. Find an equation of the ellipse with center (0, 0), one focus (4, 0), and eccentricity $\frac{2}{5}$.

Solution. Since $c = 4$ and $c/a = \frac{2}{5}$, $a = 10$. Also $b^2 = a^2 - c^2 = 100 - 16 = 84$. Therefore the required equation is

$$\frac{x^2}{100} + \frac{y^2}{84} = 1.$$

PROBLEMS

Sketch the graphs of each of the following equations (Problems 1 through 6), giving the center, foci, length of each axis, and eccentricity.

1. $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$
2. $\dfrac{x^2}{36} + \dfrac{y^2}{100} = 1$
3. $x^2 + 2y^2 = 6$
4. $4x^2 + 3y^2 = 12$
5. $36x^2 + 100y^2 = 1$
6. $9x^2 + 4y^2 = 1$

In Problems 7 through 14 find an equation of the ellipse satisfying the following conditions:

7. Ends of minor axis are $(0, \pm 2)$; distance between foci is 6.
8. Foci are $(3, 0)$ and $(-3, 0)$; length of either latus rectum is 9.
9. Vertices are $(\pm 6, 0)$; foci are $(\pm 4, 0)$.
10. Foci are $(0, \pm 2)$; eccentricity is $\frac{2}{3}$.
11. Center at the origin; one vertex is $(4, 0)$; passes through the point $(2, \frac{1}{2}\sqrt{3})$.
12. Center at the origin; major axis along x-axis; passes through the two points $(4, 3)$ and $(6, 2)$.
13. Center at the origin; symmetrical with respect to axes; passes through $(1, -4)$ and $(2, 2)$.
14. Center at the origin; symmetrical with respect to axes; length of latus rectum is $\frac{2}{3}$; eccentricity is $\frac{2}{3}$.
15. Derive Eq. (20–9), showing all steps.
16. Using Definition 20–3, determine an equation of the ellipse with foci $(2, 3)$ and $(8, 3)$ and length of major axis 10.
17. Using Definition 20–3, determine an equation of the ellipse with foci $(5, 1)$ and $(5, -3)$ and length of minor axis 6.
18. With the help of Definition 20–3, show that the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

represents an ellipse with center (h, k), foci $(h \pm c, k)$, and major axis of length $2a$.

19. What would be the conditions on the coefficients of Eq. (20–1) if this equation represented the ellipse given in Problem 18?
20. Find the locus of the centers of all circles which are tangent to the circle $(x + 1)^2 + y^2 = 36$ and pass through the point $(1, 0)$. Why is this locus an ellipse?
21. Find the locus of all points P such that the ratio of the distance of P from the point $(4, 0)$ to its distance from the line $x = 6$ is equal to $\sqrt{6}/3$.
22. Find the locus of all points P such that the ratio of the distance of P from the point $(0, \frac{3}{2})$ to its distance from the line $y = 6$ is equal to $\frac{1}{2}$.
23. Find the locus of all points P such that the ratio of the distance of P from the point $(c, 0)$ to its distance from the line $x = a^2/c$ is equal to c/a.
24. The statement of Problem 23, with $e = c/a$ $(0 \leq e < 1)$, is sometimes used as the definition of an ellipse. Show that this statement is equivalent to: The ellipse with Eq. (20–8) is the locus of all points P such that the ratio of the distance of P from the point $(ae, 0)$ to its distance from the line $x = a/e$ is equal to the eccentricity e, where $0 \leq e < 1$.
25. Find equations of the tangent and normal lines to the ellipse $6x^2 + y^2 = 100$ at $(4, 2)$.

26. As in the case of the parabola (Problem 19, Article 20–3), the line segment between the focus F and any point P on the ellipse is called a focal radius. Prove that any tangent to an ellipse makes equal angles with the focal radii drawn to the point of tangency.

27. Find the coordinates of the point $P(x, y)$ in the first quadrant if the tangent to the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ at this point has equal intercepts.

28. At what points on the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ do the tangent and normal form an isosceles triangle with the x-axis?

29. Find equations of the tangents to $9x^2 + 16y^2 = 144$ which are perpendicular to the line $4x - y + 1 = 0$.

30. Find an equation of the ellipse with center at (1, 2), axis parallel to the y-axis, the curve passing through (3, 2) and (1, 6).

20–5 THE HYPERBOLA

The hyperbola can be described geometrically as the intersection of a right circular cone and a plane inclined so that it cuts both nappes and does not pass through the vertex of the cone (see Fig. 20–15). We again define this conic section in terms of a locus of points:

20-15

Definition 20–4. A *hyperbola* is the locus of all points in the plane such that the difference of its distances from two fixed points is a constant.

The two fixed points, usually denoted by F and F', are called the *foci* of the hyperbola. The mid-point of the line segment FF' is called the *center* of the hyperbola. If the length of this line segment is denoted by $2c$ and the numerical difference of the distances (or constant mentioned in the definition) by $2a$, we have $2c > 2a$ or $c > a$. (Why?)

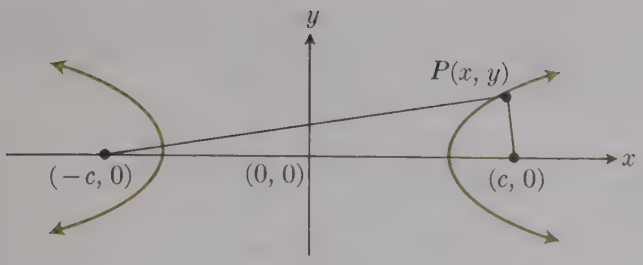
As in the case of the ellipse, we wish to choose the coordinate axes so as to obtain an equation for the general hyperbola in as simple a form as possible. If we choose the center as the origin and the foci $F(c, 0)$ and $F'(-c, 0)$ on the x-axis (Fig. 20–16), the definition requires that the coordinates of the point $P(x, y)$ satisfy the condition

$$|F'P| - |FP| = \pm 2a,$$

or

$$\sqrt{(x + c)^2 + (y - 0)^2} - \sqrt{(x - c)^2 + (y - 0)^2} = \pm 2a. \tag{20–10}$$

Since the difference mentioned in the definition may be taken in either order, the hyperbola consists of two branches. This explains the $\pm$ sign in Eq. (20–10). The plus sign holds on the right branch, the minus sign on the left.



20-16

If we repeat the process used with Eq. (20–6), we again obtain

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2). \tag{20–11}$$

But now $c > a$, so that $c^2 > a^2$, and hence Eq. (20–11) may be written

$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2), \tag{20–12}$$

where $c^2 - a^2 > 0$. Now, letting $c^2 - a^2 = b^2$ and dividing each member by a^2b^2, we have the desired result,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{20–13}$$

Since, as in the case of the ellipse, the steps in this calculation are reversible, Eq. (2–13) is the required equation. It is considered the *standard equation of the hyperbola* with its center at the origin.

By discussing the graph of this equation, as in Article 15–3, we find:

a) it is symmetric with respect to both axes and thus to the origin;
b) it has $(\pm a, 0)$ for its x-intercepts; it has no y-intercepts;
c) the curve exists for all real values of y; it exists for all x such that $|x| \geq a$.

The points $V(a, 0)$ and $V'(-a, 0)$ are called the *vertices*, and the line segment VV' of length $2a$ is called the *transverse axis*, since it crosses the curve. The other axis is called the *conjugate axis*. The chord through either focus perpendicular to the transverse axis is again called the *latus rectum;* it has a length of $2b^2/a$. The ratio c/a is again called the *eccentricity*, but $e > 1$ for any hyperbola (why?).

There is one further fact which will help sketch the graph of Eq. (20–13). Let us consider the part of the graph which lies in the first quadrant. (If we use the symmetry of the curve, its behavior in the other quadrants will also be clear.) Since

$$y = \frac{b}{a}\sqrt{x^2 - a^2}, \tag{20–14}$$

as x increases so does y, but we can be more explicit. Consider the line through the center whose equation is $y = (b/a)x$, or $bx - ay = 0$. The distance d from any point (x_1, y_1) to this line is [recall Eq. (8–18)]

$$d = \frac{|bx_1 - ay_1|}{\sqrt{a^2 + b^2}} = \frac{|bx_1 - ay_1|}{c}.$$

If the point (x_1, y_1) is on the hyperbola, its coordinates satisfy the equation, so that

$$b^2x_1^2 - a^2y_1^2 = (bx_1 - ay_1)(bx_1 + ay_1) = a^2b^2,$$

and thus

$$d = \frac{a^2b^2}{c|bx_1 + ay_1|}. \tag{20–15}$$

As x increases along the curve, so does y (Eq. 20–14), and thus

$$\lim_{x_1 \to \infty} d = \lim_{x_1 \to \infty} \frac{a^2b^2}{c|bx_1 + ay_1|} = 0.$$

Thus, as x increases indefinitely, the distance from the curve to the straight line approaches zero. Such a straight line is called an *asymptote* of the hyperbola.* In the same way, the line $y = (-b/a)x$ may be shown to be an asymptote, so that the two lines

$$\left(\frac{x}{a} + \frac{y}{b}\right)\left(\frac{x}{a} - \frac{y}{b}\right) = 0$$

or

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \tag{20–16}$$

are asymptotes of the hyperbola, Eq. (20–13). By sketching these lines and using the facts stated in (a), (b), and (c), we can easily draw the graph of Eq. (20–13).

As in the case of the ellipse, if the foci had been located on the y-axis, namely $F(0, c)$ and $F'(0, -c)$, the transverse axis would lie along the y-axis, again of length $2a$, but the equation would become

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1, \tag{20–17}$$

with the vertices $V(0, a)$ and $V'(0, -a)$. In either case, Eq. (20–13) or (20–17), $b^2 = c^2 - a^2$, and the asymptotes are the same.

Example 1. Sketch the graph of the hyperbola $9x^2 - 16y^2 = 144$, giving the coordinates of the foci and vertices, the eccentricity, length of the latus rectum, and the equations of the asymptotes.

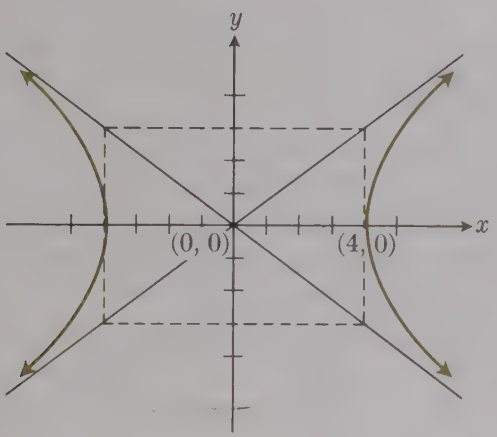
---

* Recall from Article 15–3 the definition of a horizontal or vertical asymptote and note the similarity to the definition given here.

Solution. Dividing each member by 144, we have

$$\frac{x^2}{16} - \frac{y^2}{9} = 1,$$

so that $a = 4$, $b = 3$, and $c = 5$. Since the coefficient of x^2 is positive, the transverse axis is the x-axis. Thus the foci have coordinates $(\pm 5, 0)$, the vertices $(\pm 4, 0)$, the eccentricity is $\frac{5}{4}$, and the length of the latus rectum is $2 \cdot \frac{9}{4} = \frac{9}{2}$. By replacing the constant term by zero, we have the asymptotes $(x^2/16) - (y^2/9) = 0$, or $3x + 4y = 0$ and $3x - 4y = 0$. The graph appears in Fig. 20–17.



20-17

Example 2. Find an equation of the hyperbola having $(0, \pm 6)$ for foci and $(0, \pm 4)$ for vertices.

Solution. With these coordinates for the foci and vertices, we immediately have $c = 6$ and $a = 4$. Since $b^2 = c^2 - a^2$, $b^2 = 20$. Also, the required hyperbola has y for its transverse axis, so that the required equation is

$$\frac{y^2}{16} - \frac{x^2}{20} = 1.$$

PROBLEMS

Sketch the graphs of each of the following equations (Problems 1 through 6), giving the center, foci, length of each axis, equations of the asymptotes, and eccentricity.

1. $\frac{x^2}{36} - \frac{y^2}{64} = 1$
2. $\frac{y^2}{16} - \frac{x^2}{9} = 1$
3. $3x^2 - 4y^2 = 12$
4. $5x^2 - 16y^2 = 80$
5. $9y^2 - 16x^2 = 1$
6. $4x^2 - y^2 = 1$

In Problems 7 through 14 find an equation of the hyperbola satisfying the following conditions:

7. Foci are $(\pm 6, 0)$; length of latus rectum is 18.
8. Vertices are $(0, \pm 4)$; eccentricity is $\frac{5}{2}$.
9. Equations of asymptotes are $x^2 - 4y^2 = 0$; passes through $(3, -1)$.
10. Center at origin; one focus is $(4, 0)$; eccentricity is 8.
11. Center at origin; axes along coordinate axes; latus rectum length is $\frac{4}{3}$; slopes of asymptoes are ± 3.
12. Center at origin; axes along coordinate axes; e is $\sqrt{6}/2$; passes through $(1, 2)$.
13. Center at origin; transverse axis along y-axis; length of latus rectum is $\frac{2}{3}$; passes through $(1, -2)$.
14. Center at the origin; transverse axis along x-axis; passes through $(2, -3)$ and $(-7, 6)$.
15. The hyperbola in whose equation $a = b$ is called a *rectangular hyperbola*. Show that for a rectangular hyperbola:

    a) the asymptotes intersect at right angles; b) the eccentricity is $\sqrt{2}$.
16. Using Definition 20–4, determine an equation of a hyperbola with foci $(-5, 4)$, $(1, 4)$, and length of transverse axis 4.
17. Using Definition 20–4, determine an equation of a hyperbola with vertices $(2, -3)$, $(8, -3)$, and eccentricity $\frac{5}{3}$.
18. With the help of Definition 18–4, show that the equation

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

    represents a hyperbola with center (h, k), foci $(h \pm c, k)$, and transverse axis of length $2a$.
19. What would be the conditions on the coefficients of Eq. (20–1) if this equation represented the hyperbola given in Problem 18?
20. Find the locus of the center of all circles which are tangent to the circle $(x-1)^2 + y^2 = 1$ and pass through the point $(-1, 0)$. Why is this locus a hyperbola?
21. Find the locus of all points P such that the ratio of the distance of P from the point $(0, \frac{9}{2})$ to its distance from the line $y = \frac{1}{18}$ is equal to 9.
22. Find the locus of all points P such that the ratio of the distance from the point $(6, 0)$ to its distance from the line $x = \frac{3}{2}$ is equal to 2.
23. The statement of Problem 23, Article 20–4, with $e = c/a$ (but $e > 1$), is sometimes used as the definition of a hyperbola. Show that this statement is equivalent to: The hyperbola with Eq. (20–13) is the locus of all points P such that the ratio of the distance of P from the point $(ae, 0)$ to its distance from the line $x = a/e$ is equal to the eccentricity e, where $e > 1$.
24. Prove that the product of the distances from any point on a hyperbola to the two asymptotes is a constant.
25. Find equations of the tangent and normal lines to the hyperbola $6x^2 - 4y^2 = 38$ at the point $(-3, 2)$.

26. Prove that any tangent to a hyperbola bisects the angle between the focal radii drawn to the point of tangency.

27. For what value(s) of k will the line $3x - 2y + k = 0$ be tangent to the hyperbola $x^2 - 3y^2 = 1$.

28. Find the locus of the center of all circles which pass through $(-6, 0)$ and are tangent to the circle $x^2 + y^2 - 12x = 0$.

29. Prove that the distance from a focus of a hyperbola to an asymptote is numerically equal to the semi-conjugate axis ($|b|$).

30. A point moves so that the product of its distances from the lines $y - mx = 0$ $y + mx = 0$ is equal to a constant k^2. Prove that the locus of this point is a hyperbola.

20–6 SUMMARY AND APPLICATIONS

We have seen that each of the conic sections may be defined geometrically as the intersection of a right circular cone and a plane in various positions. The definitions we gave for the conics as loci, however, made it possible to study each separately and to emphasize the graphs and properties of each. It seems appropriate at this time to unify the parabola, ellipse, and hyperbola in the light of Problem 24, Article 20–4, and Problem 23, Article 20–5.

Definition 20–5. For any fixed line l and any fixed point F not on the line l, the locus of all points P in the plane such that the ratio of the distance of P from the point F to the distance from the line l is equal to a constant $e > 0$ is

$$\begin{aligned} &\textit{an ellipse} \text{ for } e < 1, \\ &\textit{a parabola} \text{ for } e = 1, \\ &\textit{a hyperbola} \text{ for } e > 1. \end{aligned} \tag{20–18}$$

The results of the two problems mentioned above, together with the original definition of the parabola, should indicate the equivalence of Definition 20–5 with those previously used. In each case, the fixed line is called the *directrix*, and the fixed point the *focus*. In fact, as has been mentioned, both the ellipse and hyperbola have two foci, and with these, corresponding directrices. The following lists some applications of the different conics in various fields of science.

1. Parabola:
   a) The path of a projectile due to the force of gravity if air resistance is neglected.
   b) Cable of a suspension bridge if the weight of the bridge is distributed uniformly.
   c) A parabolic curve which approximates a curve through a set of points (by a method known as "least squares").
   d) Parabolic reflectors and surfaces used in telescopes, optical instruments, etc.

2. Ellipse:
   a) Orbits of the planets, including the earth, with the sun at a focus.
   b) Orbits of some comets.
   c) Elliptical gears.
   d) Arches of certain bridges.

3. Hyperbola:
   a) Certain laws in physics are hyperbolic in nature. Any two quantities that are inversely proportional are governed by a law of this type.*
   b) Certain hyperbolic lenses.
   c) Orbits of some comets.
   d) Families of curves (used to locate position in certain navigational systems).

20–7 TRANSLATION OF AXES

Throughout our discussion of analytic methods we have constantly tried to place our axes relative to the curve, locus, or figure under consideration in a position which will be to our advantage. This was emphasized in Article 8–7, and again in the last few articles. Sometimes, however, we are given an equation with a specific rectangular coordinate system. It may then greatly simplify our work if we change the system of axes. Such a change may consist of a translation of the axes, a rotation of the axes, or both.

Definition 20–6. If either or both of the two rectangular coordinate axes are moved so that the origin takes on a new position but each axis remains parallel to its original position, the change is called a *translation of the axes.*

Let us be more specific. If any fixed point whose original coordinates are (h, k) is denoted by O', and the two lines parallel to the x- and y-axes through O' are regarded as the new x'- and y'-axes respectively, any point P with coordinates (x, y), relative to the original axes, will also have coordinates (x', y') relative to the new axes.

Theorem 20–3. As shown in Fig. 20–18, the relation between the new and the old coordinates for this translation is given by

$$x' = x - h, \qquad y' = y - k, \tag{20–19}$$

or

$$x = x' + h, \qquad y = y' + k. \tag{20–20}$$

Proof. Figure 20–18 may be used in the proof of Eq. (20–19). Equation (20–20) follows directly from (20–19).

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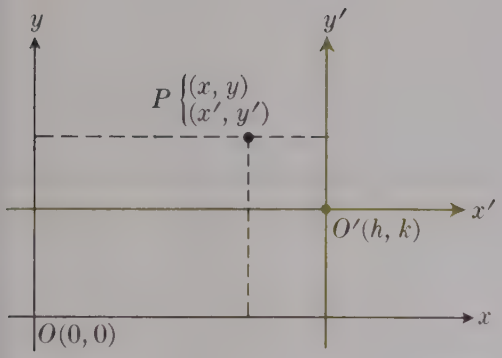
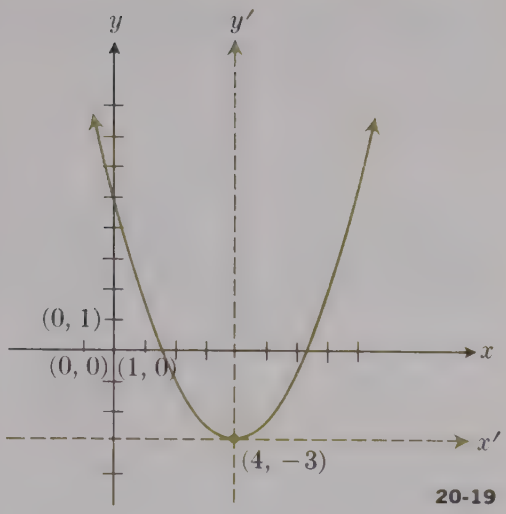
* Article 20–8, where we take up the graph of $xy = k$, makes clear the fact that inverse proportion is hyperbolic.



20-18



20-19

With the use of Eq. (20–20), any equation in x and y may be transformed to an equation in x' and y'. Graphically, the curve of the equation in the xy-coordinate system will coincide with that of the new equation in the $x'y'$-coordinate system; that is, the axes are translated, not the curve.

Example 1. Find an equation relative to the new set of axes with the new origin located at $(4, -3)$ if the original equation is $x^2 - 8x - 2y + 10 = 0$. Sketch the curve showing both sets of axes.

Solution. Any point $P(x, y)$ whose coordinates (x, y) satisfy $x^2 - 8x - 2y + 10 = 0$ has for its new coordinates $x' = x - 4$ and $y' = y + 3$, since $h = 4$ and $k = -3$. Thus, $x = x' + 4$ and $y = y' - 3$ gives the original coordinates in terms of the new ones. Substituting these values for x and y in the given equation, we have

$$(x' + 4)^2 - 8(x' + 4) - 2(y' - 3) + 10 = 0,$$

the required new equation. Simplifying this, we get

$$x'^2 + 8x' + 16 - 8x' - 32 - 2y' + 6 + 10 = 0,$$

or

$$x'^2 = 2y'.$$

The graph appears in Fig. 20–19 with both sets of axes shown.

In Example 1, the new equation $x'^2 = 2y'$ is much simpler than the original equation $x^2 - 8x - 2y + 10 = 0$. Hence we see one of the important uses of translation of axes, the simplification of the graphing of a given curve.

Example 2. Determine the translation to eliminate the x- and y-terms in the equation $4x^2 + 9y^2 - 16x + 18y - 11 = 0$, and find the resulting equation relative to this translation.

Solution 1. Let us complete the square (recall Article 8–9, or Example 1, Article 20–1). We have

$$4x^2 + 9y^2 - 16x + 18y - 11 = 0,$$

or

$$4(x^2 - 4x \quad) + 9(y^2 + 2y \quad) = 11.$$

By adding 4 and 1, inside the first and second parentheses, and therefore 25 to the right member, we have not changed the equation, but may write it in the form $4(x - 2)^2 + 9(y + 1)^2 = 36$ or, dividing the members by 36,

$$\frac{(x - 2)^2}{9} + \frac{(y + 1)^2}{4} = 1.$$

If we let $x' = x - 2$ and $y' = y + 1$, so that $h = 2$ and $k = -1$, the equation relative to the new coordinate system is

$$\frac{x'^2}{9} + \frac{y'^2}{4} = 1,$$

which has no x- or y-terms. This method of choosing a translation is frequently used to facilitate the graphing of conic sections when their axes are parallel to the coordinate axes.

Solution 2. The equation relative to the new axes can be obtained for any translation by substituting $x = x' + h$ and $y = y' + k$ in $4x^2 + 9y^2 - 16x + 18y - 11 = 0$. Thus,

$$4(x' + h)^2 + 9(y' + k)^2 - 16(x' + h) + 18(y' + k) - 11 = 0.$$

Simplifying, we have

$$4x'^2 + 8hx' + 4h^2 + 9y'^2 + 18ky' + 9k^2 - 16x' - 16h + 18y' + 18k - 11 = 0,$$

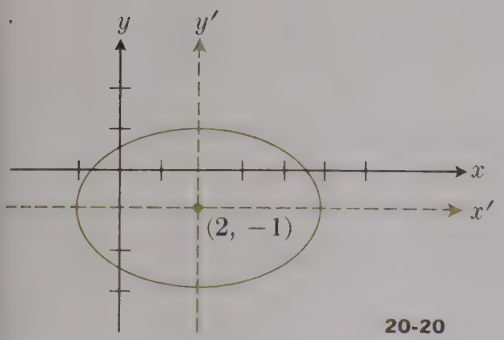
and then collecting terms, we obtain

$$4x'^2 + 9y'^2 + (8h - 16)x' + (18k + 18)y' + 4h^2 + 9k^2 - 16h + 18k - 11 = 0.$$

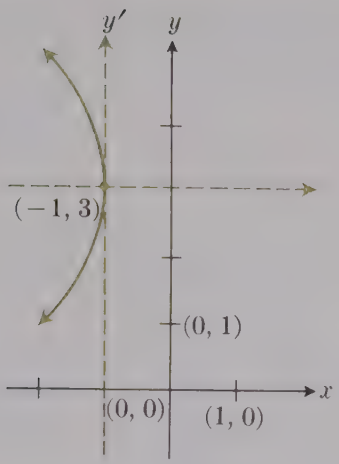
Since we desire to eliminate the x'- and y'-terms, we must have their coefficients zero, so that $8h - 16 = 0$ and $18k + 18 = 0$ or $h = 2$ and $k = -1$. Thus the required transformation is $x = x' + 2$ and $y = y' - 1$ (or $x' = x - 2$, $y' = y + 1$, as was found in Solution 1). If these values for x and y are substituted in the original equation, we obtain as before

$$\frac{x'^2}{9} + \frac{y'^2}{4} = 1.$$

The graph of the equations with both coordinate systems is shown in Fig. 20–20.



20-20



20-21

Example 3. Find the coordinates of the vertex, focus, equation of directrix, and length of latus rectum of the parabola

$$y^2 + 4x - 6y + 13 = 0.$$

Solution. By completing the square, we have

$$y^2 - 6y + 9 = -4x - 13 + 9 = -4x - 4,$$

or

$$(y - 3)^2 = -4(x + 1).$$

By letting $y' = y - 3$ and $x' = x + 1$, we have

$$y'^2 = -4x',$$

so that the new origin, and vertex, is $(-1, 3)$ in the original system. Therefore, in the original coordinate system, since $4p = -4$ or $p = -1$, the vertex is $(-1, 3)$, the focus is $(-2, 3)$, the equation of the directrix is $x = 0$, and the length of the latus rectum is 4. Figure 20–21 shows the graph of the given parabola.

PROBLEMS

In Problems 1 through 4 transform the given equation by translating the coordinate axes to the new origin indicated.

1. $2x - 3y + 5 = 0$; $(2, 3)$
2. $x^2 + y^2 - 2x + 6y + 6 = 0$; $(1, -3)$
3. $x^2 - 4y^2 + 6x + 8y + 1 = 0$; $(-3, 1)$
4. $xy - 2y + 3x - 7 = 0$; $(2, -3)$

In Problems 5 through 10 transform the given equation into another equation lacking terms of the first degree, by a translation of the coordinate axes.

5. $x^2 - y^2 - 4x + 4y - 1 = 0$
6. $4x^2 + 9y^2 - 24x + 36y + 36 = 0$
7. $xy - 7x + 3y - 22 = 0$
8. $4x^2 + y^2 + 16x - 10y + 37 = 0$
9. $2y^2 + 4y - 8x + 18 = 0$
10. $9x^2 - 4y^2 + 36x + 8y - 4 = 0$

In Problems 11 through 15 find the coordinates of the center (if any), vertices, and foci, the eccentricity, and equations of asymptotes (if any), and sketch the curve.

11. $x^2 + 4y^2 - 10x - 40y + 109 = 0$
12. $9x^2 - 24x + 72y + 16 = 0$
13. $9x^2 - 4y^2 - 54x + 8y + 113 = 0$
14. $5x^2 - 4y^2 - 20x - 24y - 36 = 0$
15. $12x^2 + 16y^2 - 12x - 9 = 0$
16. Do Problem 17, Article 20–3, by the method of this article.
17. Do Problem 18, Article 20–4, and Problem 18, Article 20–5, by the method of this article.
18. Note that the graph of Eq. (20–1) with $B = 0$, if it exists and does not consist of just a point or straight line, is one of the conic sections. What are the conditions on the coefficients for a circle, a parabola, an ellipse, and a hyperbola?

20–8 ROTATION OF AXES

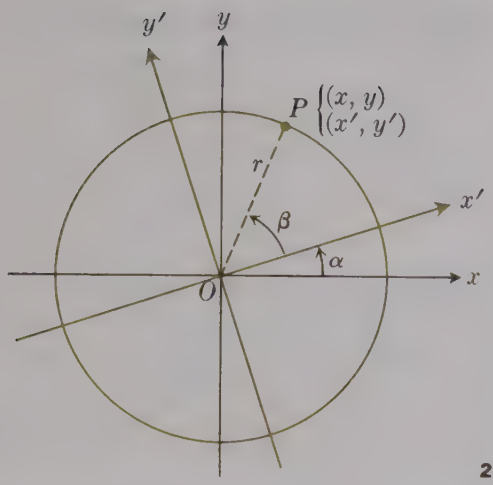
Often the rotation of axes is an advantageous procedure.

Definition 20–7. If, in a rectangular coordinate system with given x- and y-axes, a new set of rectangular coordinate axes x' and y' are chosen having the same origin, the change is called *a rotation of axes.*

As in the case of a translation of the coordinate axes to a new set of axes, so in the case of rotation of a set of coordinate axes to a new set of axes, there exists a relation between the original coordinates (x, y) of any point P and the new set of coordinates (x', y') of this same point. If α denotes the angle from the original positive x-axis to the new positive x'-axis, and the coordinates of any point P are denoted before and after the rotation (Fig. 20–22) by (x, y) and (x', y'), respectively, we have the following theorem.

Theorem 20–4. The relation between the new and the old coordinates, if the axes are rotated through an angle α, is given by

$$\begin{aligned} x &= x' \cos \alpha - y' \sin \alpha, \\ y &= x' \sin \alpha + y' \cos \alpha. \end{aligned} \tag{20–21}$$



20-22

Proof. In Fig. 20–22 the length of the line between the origin and any point P is denoted by r. The point P has coordinates (x, y) in the original system and (x', y') in the new system, where the angle from the x-axis to the x'-axis is α, and a positive angle is measured in a counterclockwise direction. Let β be the angle from the x'-axis to the line OP. Then the x- and y-coordinates of P may be expressed as follows:

$$\begin{aligned} x &= r\cos(\alpha + \beta), \\ y &= r\sin(\alpha + \beta) \end{aligned} \tag{20–22}$$

or, recalling Eqs. (6–29), (6–30),

$$\begin{aligned} x &= r\cos\alpha\cos\beta - r\sin\alpha\sin\beta, \\ y &= r\sin\alpha\cos\beta + r\cos\alpha\sin\beta. \end{aligned} \tag{20–23}$$

Since

$$x' = r\cos\beta, \qquad y' = r\sin\beta,$$

Eq. (20–23) becomes

$$x = x'\cos\alpha - y'\sin\alpha$$

and

$$y = x'\sin\alpha + y'\cos\alpha.$$

To express x' and y' in terms of x and y, Eq. (20–21) may be solved for x' and y'. We recall Problem 17, Article 8–5, and obtain

$$\begin{aligned} x' &= x\cos\alpha + y\sin\alpha, \\ y' &= -x\sin\alpha + y\cos\alpha. \end{aligned} \tag{20–24}$$

Example 1. If the original coordinate axes are rotated through an angle of 45°, find the transformed equation, relative to the $x'y'$-system, of the original curve whose equation is $y^2 - x^2 = 2$.

Solution. Since $\alpha = 45°$, the necessary transformation becomes

$$x = \frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y',$$

$$y = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'.$$

Substituting these values in $y^2 - x^2 = 2$, we have

$$\frac{(x' + y')^2}{2} - \frac{(x' - y')^2}{2} = 2$$

or, upon simplifying, $x'y' = 1$. The curve is shown in Fig. 20–23 with both sets of coordinate axes.

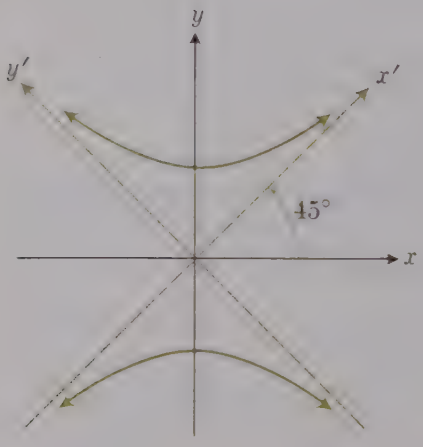
We recall the difficulty involved in graphing certain curves when xy-terms were present. One of the distinct advantages of a rotation of axes is the possibility of elimination of such terms. We may then sketch the simpler curve in the new rotated system, and thus obtain the graph of the curve in the original system. Consider the next example with this in mind.

Example 2. Sketch the graph of the equation $3x^2 + \sqrt{3}xy + 2y^2 = 4$ by rotating the coordinate axes through an angle of 30° and using the equation relative to the new axes.

Solution. Since $\alpha = 30°$, we have, by using Eq. (20–21),

$$x = \frac{\sqrt{3}}{2}x' - \frac{1}{2}y',$$

$$y = \frac{1}{2}x' + \frac{\sqrt{3}}{2}y',$$



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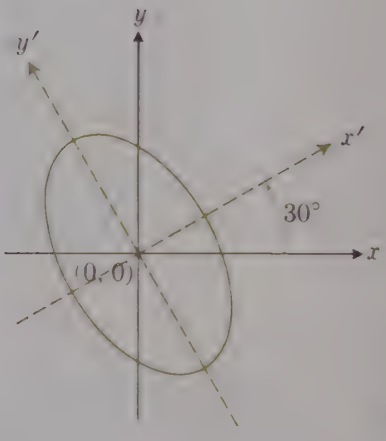


20-24

which represents the proper transformation. Substituting these values in

$$3x^2 + \sqrt{3}\,xy + 2y^2 = 4,$$

we have

$$\frac{3(\sqrt{3}\,x' - y')^2}{4} + \frac{\sqrt{3}\,(\sqrt{3}\,x' - y')(x' + \sqrt{3}\,y')}{4} + \frac{2(x' + \sqrt{3}\,y')^2}{4} = 4,$$

or

$$3(3x'^2 - 2\sqrt{3}\,x'y' + y'^2) + \sqrt{3}\,(\sqrt{3}\,x'^2 + 2x'y' - \sqrt{3}\,y'^2) + 2(x'^2 + 2\sqrt{3}\,x'y' + 3y'^2) = 16.$$

This equation reduces to

$$7x'^2 + 3y'^2 = 8.$$

This is the equation of the same curve, relative to the new rotated axes, and is easily sketched on the $x'y'$-axes, since it represents an ellipse in standard position. See Fig. 20–24.

This example illustrates the simplification of an equation by the proper rotation. Fortunately, such a simplification, the elimination of the xy-term, can always be obtained for any equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \tag{20–1}$$

Hence the graph of the equation can be recognized and sketched.

Theorem 20–5. Given any equation in the form of Eq. (20–1), there exists an angle α, where $\tan 2\alpha = B/(A - C)$ if $A \neq C$ or $\alpha = 45°$ if $A = C$, such that the graph of this equation relative to the new set of axes obtained by rotating through the angle α has a new equation with no $x'y'$-term.

Proof. In the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

let us replace x and y by the corresponding values from Eq. (20–21), where the axes are rotated through an angle α. We get

$$A(x' \cos\alpha - y' \sin\alpha)^2 + B(x' \cos\alpha - y' \sin\alpha)(x' \sin\alpha + y' \cos\alpha) + C(x' \sin\alpha + y' \cos\alpha)^2 + D(x' \cos\alpha - y' \sin\alpha) + E(x' \sin\alpha + y' \cos\alpha) + F = 0.$$

If we collect the various like terms, we can easily see that this equation may be written in the form

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F = 0, \tag{20–25}$$

where
$$\begin{aligned} A' &= A\cos^2\alpha + B\sin\alpha\cos\alpha + C\sin^2\alpha, \\ B' &= 2(C - A)\sin\alpha\cos\alpha + B(\cos^2\alpha - \sin^2\alpha), \\ C' &= A\sin^2\alpha - B\sin\alpha\cos\alpha + C\cos^2\alpha, \\ D' &= D\cos\alpha + E\sin\alpha, \\ E' &= E\cos\alpha - D\sin\alpha. \end{aligned}$$

This expression, Eq. (20–25), is the new equation of the same curve, but relative to the new set of axes. It will have no $x'y'$-term if we choose α so that $B' = 0$, that is, so that

$$2(C - A)\sin\alpha\cos\alpha + B(\cos^2\alpha - \sin^2\alpha) = 0. \tag{20–26}$$

Recalling Eqs. (6–43), (6–44), we may write

$$(C - A)\sin 2\alpha + B\cos 2\alpha = 0$$

If $A - C = 0$, $\cos 2\alpha = 0$, so that the choice $\alpha = 45°$ will necessitate $B' = 0$. If $A \neq C$, we have

$$\tan 2\alpha = \frac{B}{A - C}. \tag{20–27}$$

Since this can always be solved for 2α, and hence for α, such an angle always exists.

Although there are infinitely many values for α (differing from one another by multiples of $\pi/2$) which would satisfy Eq. (20–26) and thus make $B' = 0$, we shall ordinarily choose the smallest positive value.

To obtain the form of Eq. (20–21) for a specific correct rotation, that is, the value of $\sin\alpha$ and $\cos\alpha$ when $\tan 2\alpha$ is given, we must first find $\cos 2\alpha$ and then use Eqs. (6–49), (6–51) in the form

$$\sin\alpha = \sqrt{\frac{1 - \cos 2\alpha}{2}}, \qquad \cos\alpha = \sqrt{\frac{1 + \cos 2\alpha}{2}}. \tag{20–28}$$

This will be illustrated in Example 3.

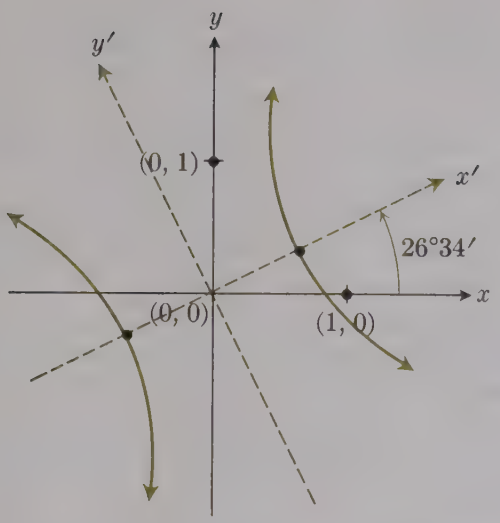
Example 3. Sketch the graph of the equation $7x^2 + 12xy - 2y^2 = 5$.

Solution. We rotate the axes through an angle α, determined by Eq. (20–27). Thus

$$\tan 2\alpha = \frac{12}{7 - (-2)} = \frac{4}{3}.$$

Since $\tan 2\alpha = \frac{4}{3}$, $\cos 2\alpha = \frac{3}{5}$. (Why?) Therefore,

$$\sin\alpha = \sqrt{\frac{1 - \frac{3}{5}}{2}} = \frac{1}{\sqrt{5}} \quad \text{and} \quad \cos\alpha = \sqrt{\frac{1 + \frac{3}{5}}{2}} = \frac{2}{\sqrt{5}},$$



20-25

and Eq. (20–21) becomes

$$x = \frac{2}{\sqrt{5}}x' - \frac{1}{\sqrt{5}}y', \qquad y = \frac{1}{\sqrt{5}}x' + \frac{2}{\sqrt{5}}y'.$$

Substituting these values in $7x^2 + 12xy - 2y^2 = 5$, we have

$$\frac{7(2x' - y')^2}{5} + \frac{12(2x' - y')(x' + 2y')}{5} - \frac{2(x' + 2y')^2}{5} = 5$$

or, simplifying,

$$2x'^2 - y'^2 = 1.$$

This hyperbola may now be sketched (Fig. 20–25) in the $x'y'$-coordinate system and the original axes shown, and we realize that the angle of rotation α is approximately 26°34′.

PROBLEMS

Sketch the graph of each of the following equations, showing each pair of coordinate axes.

1. $xy = 8$

2. $2x^2 + 4xy + 5y^2 = 2$

3. $3x^2 - 2xy + 3y^2 = 2$

4. $4xy + 3y^2 = 4$

5. $5x^2 + 4xy + 8y^2 = 36$

6. $x^2 + 36xy - 14y^2 = 52$

7. $9y^2 + 24xy + 16y^2 - 20x + 15y = 0$

8. $xy + 3x + 2y - 7 = 0$

9. $3x^2 + 2xy + 3y^2 + 16x - 16y + 52 = 0$

10. $4x^2 - 4xy + y^2 - 6x + 3y = 4$
11. $3x^2 - 4xy + 6y^2 + 8x + 4y + 10 = 0$
12. $8x^2 + 12xy + 17y^2 - 28x - 46y + 17 = 0$
13. Prove that the equation of any circle with center at the origin is unchanged under a rotation through any angle α.
14. Show that if Eq. (20–1) is transformed by a rotation through any angle α to the equation given by Eq. (20–25), then $A' + C' = A + C$. *Hint:* Use the values for A' and C' given in the proof of Theorem 20–5.
15. Under the same conditions stated in Problem 14, prove that $B'^2 - 4A'C' = B^2 - 4AC$. Quantities such as $A + C$ and $B^2 - 4AC$ which do not change their values are called *invariants* under rotation of axes.
16. Show that the graph of Eq. (20–1), if its curve is not degenerate,* is an ellipse or circle, parabola, or hyperbola according as $B^2 - 4AC$ is negative, zero, or positive. *Hint:* Use the result of Problem 15.
17. Using the results of Problem 16, state which conic section is represented by the equations given in Problems 2 through 12.

REVIEW PROBLEMS

For each of the following conics in Problems 1 through 6, find the coordinates of the center (if any), vertices, and foci, the eccentricity, and the equations of the directrices and asymptotes (if any). Sketch the curve.

1. $x^2 - 8y - 6x + 17 = 0$
2. $x^2 + 4y^2 - 10x + 16y + 37 = 0$
3. $x^2 - y^2 + 4x + 8y - 11 = 0$
4. $9y^2 - 4x^2 + 36y + 8x - 4 = 0$
5. $2y^2 - 3x + 4y + 2 = 0$
6. $9x^2 + 4y^2 - 18x + 16y - 11 = 0$
7. Find the least distance between the circle $x^2 + y^2 - 4x - 10y + 13 = 0$ and the point $(8, -3)$.
8. Find the length of the tangent to the circle $x^2 + y^2 - 4x - 10y + 13 = 0$ from the point $(8, -3)$.
9. A circle is determined by the three midpoints of the three sides of any triangle. Show that this circle also passes through the feet of the altitudes of this triangle.
10. The base of a triangle joins the points $(a, 0)$ and $(-a, 0)$, where $a > 0$. The opposite vertex moves in such a way that the difference between the tangents of the base angles is unity. Find and identify the locus of this vertex.

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* By "degenerate" we mean no curve, a point, or one or two straight lines.

11. The base of a triangle joins the points $(a, 0)$ and $(-a, 0)$, where $a > 0$. The opposite vertex moves in such a way that the sum of the tangents of the base angles is 1. Find and identify the locus of this vertex.

12. Show that for any hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the distance from a focus to an asymptote is equal to b.

13. Prove that the two parabolas with the same axis and focus, but opposite in direction, intersect at right angles.

14. Show that the two conics $2x^2 - y^2 = 2$ and $2x^2 + 3y^2 = 18$ intersect at right angles.

15. Find the locus of the centers of all circles which pass through $(3, 0)$ and are tangent to the circle $x^2 + y^2 = 25$.

16. Find the locus of the centers of all circles which pass through $(7, 0)$ and are tangent to the circle $x^2 + y^2 = 25$.

17. Find the locus of the centers of all circles which are tangent to the line $x = 3$ and to the circle $x^2 + y^2 = 25$.

18. Find the locus of the centers of all circles which are tangent to the line $x = 7$ and to the circle $x^2 + y^2 = 25$.

19. Find equations of the tangent and normal to the equation

$$2x^2 + 2y^2 - x - 5y - 8 = 0 \quad \text{at} \quad (-\tfrac{1}{2}, -1).$$

20. What is the relation between the coefficients of the conics

$$ax^2 + by^2 = 1 \quad \text{and} \quad cx^2 + dy^2 = 1$$

if they meet at right angles?

21. Find equations of the tangents to the parabola $y^2 - 2y - 8x + 9 = 0$ through $(-1, 4)$.

22. Find the points of tangency of the line with slope $-\frac{1}{2}$ which is tangent to the ellipse $x^2 + 4y^2 = 8$.

23. A circle passes through each of the foci of an ellipse and is tangent to the ellipse at two other points. What is the eccentricity of the ellipse?

24. The line $x - y + 4 = 0$ is tangent to the parabola $y^2 = 16x$. Find the coordinates of the point of tangency.

25. Explain why the locus of the centers of all circles which pass through a fixed point and are tangent to a fixed line, is a parabola.

ANSWERS TO CHAPTER 20

Article 20-2

1. (a) $(x-2)^2+(y-3)^2=1$ (b) $(x+6)^2+(y-4)^2=36$

(c) $(x+\frac{2}{3})^2+(y-\frac{3}{2})^2=\frac{1}{16}$ (d) $(x-2\sqrt{3})^2+(y+\sqrt{2})^2=5$

2. (a) Center $(-2, 3)$; radius 2 (b) Center $(3, -4)$; radius 1

(c) Center $(-4, 7)$; radius 0 (point circle)

(d) Center $(3/2, -7/2)$; radius $\sqrt{57/2}$

(e) Center $(2, -3)$; radius $\sqrt{57/2}$ (f) Center $(2/3, 1/2)$; radius $1/2$

3. $x^2+y^2=41$ **4.** $(x+3)^2+(y-2)^2=52$

5. $(x-2)^2+(y+2)^2=52$ **6.** $(x+1)^2+(y-4)^2=25$

7. $(x-3)^2+(y-2)^2=1/10$ **8.** $(x+4)^2+(y-1)^2=144/13$

9. $(x-1)^2+(y+1)^2=20$ **10.** $(x-\frac{13}{2})^2+(y-\frac{5}{2})^2=5/2$

11. $(x-1)^2+(y-5)^2=25$ **12.** $(x-3)^2+(y+2)^2=25$

13. $(x-\frac{19}{6})^2+(y-\frac{23}{3})^2=1445/36$ **14.** $(x-9)^2+(y-7)^2=100$

15. $(x-6)^2+(y-2)^2=26$ **16.** $(x+2)^2+(y-1)^2=25$

17. $(-3, -3)$ and $(-1, 1)$ **18.** $(2, -3)$ and $(-1, -1)$

21. $x-2y-4=0$. Yes **22.** They are parallel.

24. $(2, 4)$ **26.** $x+y=8$; $(6, 2)$, $(1, 7)$; $5\sqrt{2}$

27. $(x-6)^2+(y-1)^2=25$; $(x+1)^2+(y-2)^2=25$

28. $(x-1)^2+(y-1)^2=20$; $(x+\frac{13}{2})^2+(y-\frac{19}{4})^2=\frac{14485}{16}$

29. $(x-6)^2+(y+6)^2=36$; $(x-126)^2+(y+126)^2=174^2$

30. Center $(3\pm\sqrt{15}, 2\pm\sqrt{15})$

31. $(x-3)^2+(y-3)^2=5$; $(x+1)^2+(y-1)^2=5$

32. (a) $x+3y-10=0$ (b) $5x+y-12=0$

(c) $x=0$ (d) $3x-4y+7=0$

33. $y=3x$; $y=2x$; $\theta=8°$ **35.** $2x-3y+13=0$; $3x+2y-13=0$

36. $3x-y-2=0$; $x+3y-14=0$

Article 20-3

1. Focus $(2, 0)$; directrix $x=-2$; length of latus rectum 8

2. Focus $(0, 9/4)$; directrix $y=-9/4$; length of latus rectum 9

3. Focus $(0, -1/4)$; directrix $y=1/4$; length of latus rectum 1

4. Focus $(-6, 0)$; directrix $x=6$; length of latus rectum 24

5. Focus $(0, -3/8)$; directrix $y=3/8$; length of latus rectum $3/2$

6. Focus $(-7/12, 0)$; directrix $x = 7/12$; length of latus rectum $7/3$

7. $y^2 = 8x$

8. $x^2 = -12y$

9. $y^2 = -6x$

10. $x^2 = -7y$

11. $2y^2 = 9x$

12. $4x^2 = 9y$

13. $x^2 - 2xy + y^2 - 8x - 8y = 0$

14. $x^2 - 2xy + y^2 + 4x + 4y - 4 = 0$

15. $(y - 4)^2 = 2x - 5$

16. $(x - 2)^2 = 4y + 16$

18. $A = B = 0$

20. $x^2 - 2y - 1 = 0$; $x^2 - 10y - 25 = 0$

21. $y^2 + 12x - 36 = 0$; $y^2 - 4x - 4 = 0$

22. Tangent: $x - y + 2 = 0$
Normal: $x + y + 6 = 0$

23. $3x - 4y + 13 = 0$; $3x - 10y + 55 = 0$

Article 20-4

1. Center $(0, 0)$; foci $(\pm 3, 0)$; major 10, minor 8; eccentricity $3/5$

2. Center $(0, 0)$; foci $(0, \pm 8)$; major 20, minor 12; eccentricity $4/5$

3. Center $(0, 0)$; foci $(\pm\sqrt{3}, 0)$; major $2\sqrt{6}$, minor $2\sqrt{3}$; eccentricity $1/\sqrt{2}$

4. Center $(0, 0)$; foci $(0, \pm 1)$; major 4; minor $2\sqrt{3}$; eccentricity $1/2$

5. Center $(0, 0)$; foci $(\pm 2/15, 0)$; major $1/3$; minor $1/5$; eccentricity $4/5$

6. Center $(0, 0)$; foci $(0, \pm\sqrt{5}/6)$; major 1; minor $2/3$; eccentricity $\sqrt{5}/3$

7. $\dfrac{x^2}{13} + \dfrac{y^2}{4} = 1$

8. $\dfrac{x^2}{36} + \dfrac{y^2}{27} = 1$

9. $\dfrac{x^2}{36} + \dfrac{y^2}{20} = 1$

10. $\dfrac{x^2}{5} + \dfrac{y^2}{9} = 1$

11. $\dfrac{x^2}{16} + y^2 = 1$

12. $\dfrac{x^2}{52} + \dfrac{y^2}{13} = 1$

13. $\dfrac{x^2}{5} + \dfrac{y^2}{20} = 1$

14. $25x^2 + 45y^2 = 9$ (two answers)

16. $\dfrac{(x-5)^2}{25} + \dfrac{(y-3)^2}{16} = 1$

17. $\dfrac{(x-5)^2}{9} + \dfrac{(y+1)^2}{13} = 1$

19. A, C same sign; $B = 0$

20. $8x^2 + 9y^2 = 72$

21. $x^2 + 3y^2 = 24$

22. $4x^2 + 3y^2 = 27$

23. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

25. Tangent: $12x + y - 50 = 0$
Normal: $x - 12y + 20 = 0$

27. $x = \dfrac{a^2}{\sqrt{a^2 + b^2}}$, $y = \dfrac{b^2}{\sqrt{a^2 + b^2}}$

28. $x = \pm\dfrac{a^2}{\sqrt{a^2 + b^2}}$, $y = \pm\dfrac{b^2}{\sqrt{a^2 + b^2}}$

29. $x + 4y = \pm 4\sqrt{10}$

30. $\dfrac{(x-1)^2}{4} + \dfrac{(y-2)^2}{16} = 1$

Article 20-5

1. Center (0, 0); foci $(\pm 10, 0)$; transverse 12, conjugate 16; eccentricity 5/3; asymptotes $y = \pm\frac{4}{3}x$
2. Center (0, 0); foci $(0, \pm 5)$; transverse 8, conjugate 6; eccentricity 5/4; asymptotes $y = \pm\frac{3}{4}x$
3. Center (0, 0); foci $(\pm\sqrt{7}, 0)$; transverse 4, conjugate $2\sqrt{3}$; eccentricity $\sqrt{7}/2$; asymptotes $y = \pm(\sqrt{3}/2)x$
4. Center (0, 0); foci $(\pm\sqrt{21}, 0)$; transverse 8, conjugate $2\sqrt{5}$; eccentricity $\sqrt{21}/4$ asymptotes $y = \pm(\sqrt{5}/4)x$
5. Center (0, 0); foci $(0, \pm 5/12)$; transverse 2/3, conjugate 1/2; eccentricity 5/4; asymptotes $y = \pm\frac{3}{4}x$
6. Center (0, 0); foci $(\pm\sqrt{5}/2, 0)$; transverse 1, conjugate 2; eccentricity $\sqrt{5}$; asymptotes $y = \pm 2x$

7. $\frac{x^2}{9} - \frac{y^2}{27} = 1$
8. $\frac{y^2}{16} - \frac{x^2}{84} = 1$
9. $x^2 - 4y^2 = 5$
10. $252x^2 - 4y^2 = 63$
11. $729x^2 - 81y^2 = 4$
12. $y^2 - 2x^2 = 2$
13. $y^2 - 3x^2 = 1$
14. $5y^2 - 3x^2 = 33$
16. $\frac{(x+2)^2}{4} - \frac{(y-4)^2}{5} = 1$
17. $\frac{(x-5)^2}{9} - \frac{(y+3)^2}{16} = 1$
19. A, C opposite signs; $B = 0$
20. $12x^2 - 4y^2 = 3$
21. $80y^2 - x^2 = 20$
22. $3x^2 - y^2 = 27$
25. Tangent: $9x + 4y + 19 = 0$; normal: $4x - 9y + 30 = 0$

Article 20-7

1. $2x' - 3y' = 0$
2. $x'^2 + y'^2 = 4$
3. $x'^2 - 4y'^2 = 4$
4. $x'y' = 1$
5. $x'^2 - y'^2 = 1$ (where $x' = x - 2$; $y' = y - 2$)
6. $4x'^2 + 9y'^2 = 36$ (where $x' = x - 3$; $y' = y + 2$)
7. $x'y' = 1$ (where $x' = x + 3$, $y' = y - 7$)
8. $4x'^2 + y'^2 = 4$ (where $x' = x + 2$, $y' = y - 5$)
9. $y'^2 = 4x'$ (where $x' = x - 2$, $y' = y + 1$) cannot eliminate the x term
10. $9x'^2 - 4y'^2 = 36$ (where $x' = x + 2$, $y' = y - 1$)
11. Center (5, 5); vertices $(5 \pm 4, 5)$; foci $(5 \pm 2\sqrt{3}, 5)$; eccentricity $\sqrt{3}/2$
12. Vertex (4/3, 0); focus $(4/3, -2)$; eccentricity 1
13. Center (3, 1); vertices $(3, 1 \pm 3)$; foci $(3, 1 \pm \sqrt{13})$; eccentricity $\sqrt{13}/3$; asymptotes $4x'^2 - 9y'^2 = 0$
14. Center $(2, -3)$; vertices $(2 \pm 2, -3)$; foci $(2 \pm 3, -3)$; eccentricity 3/2; asymptotes $5x'^2 - 4y'^2 = 0$
15. Center (1/2, 0); vertices $(1/2 \pm 1, 0)$; foci $(1/2 \pm 1/2, 0)$; eccentricity 1/2
18. Circle $A = C$, $B = 0$; Parabola $A = B = 0$ *or* $B = C = 0$; Ellipse A, C same sign; Hyperbola A, C different sign *or* $A = C = 0$, $B \neq 0$.

Article 20-8

The new equations when the axes are rotated become:

1. $x'^2 - y'^2 = 16$, $\theta = 45°$
2. $6x'^2 + y'^2 = 2$, $\tan 2\theta = -4/3$
3. $x'^2 + 2y'^2 = 1$, $\theta = 45°$
4. $4x'^2 - y'^2 = 4$, $\tan 2\theta = -4/3$
5. $9x'^2 + 4y'^2 = 36$, $\tan 2\theta = -4/3$
6. $x'^2 - 2y'^2 = 4$, $\tan 2\theta = 12/5$
7. $x'^2 = -y'$, $\tan 2\theta = -24/7$
8. $\left(x' + \frac{5}{\sqrt{2}}\right)^2 - \left(y' + \frac{1}{\sqrt{2}}\right)^2 = 26$, $\theta = 45°$
9. $2x'^2 + y'^2 - 8\sqrt{2}\,y' + 26 = 0$, $\theta = 45°$
10. $5x'^2 - 3\sqrt{5}\,x' = 4$, $\tan 2\theta = -4/3$
11. $2(x' + \sqrt{5})^2 + 7y'^2 = 0$, $\tan 2\theta = 4/3$; point
12. $4\left(x' - \frac{3}{\sqrt{5}}\right)^2 + \left(y' + \frac{1}{\sqrt{5}}\right)^2 = 4$, $\tan 2\theta = -4/3$

REVIEW PROBLEMS

Chapter 20

1. Vertex (3, 1), focus (3, 3), eccentricity 1, equation of directrix, $y = -1$
2. Center (5, -2), vertices (3, -2), (7, -2), foci ($5 \pm \sqrt{3}$, -2), eccentricity $\sqrt{3}/2$, directrices equations $x = 5 \pm 4/\sqrt{3}$
3. Center (-2, 4), vertices (-2, 4 ± 1), foci (-2, $4 \pm \sqrt{2}$), eccentricity $\sqrt{2}$, directrices equations, $y = 4 \pm \frac{1}{\sqrt{2}}$, asymptote equations, $x + y - 2 = 0$, $x - y + 6 = 0$.
4. Center (1, -2), vertices (1, -2 ± 2), foci (1, $-2 \pm \sqrt{13}$), eccentricity $\sqrt{13}/2$, directrices equations $y = -2 \pm 4/\sqrt{13}$, asymptote equations $3x - 2y - 5 = 0$, $3x + 2y = 0$
5. Vertex (-1, 0), focus ($-\frac{5}{8}$, 0), eccentricity 1, equation of directrix $x = -11/8$
6. Center (1, 2), vertices (1, 2 ± 3), foci (1, 2 ± 2), eccentricity $\sqrt{5}/3$, directices equations $y = 2 \pm 9/\sqrt{5}$
7. 6
8. $2\sqrt{21}$
10. Hyperbola, $x^2 + 2xy - a^2 = 0$
11. Hyperbola, $x^2 - 2xy - a^2 = 0$
15. Ellipse, $16x^2 + 25y^2 - 48x - 64 = 0$
16. Hyperbola, $24x^2 - 25y^2 + 182x - 1081 = 0$
17. $y^2 = 4x + 4$, $y^2 = 64 - 16x$
18. $y^2 = 144 - 24x$, $y^2 = 4 - 4x$
19. Tangent, $2x + 6y + 7 = 0$; normal $6x - 2y + 1 = 0$
20. $\frac{ab}{a - b} = \frac{cd}{c - d}$
21. Tangent $4x - 3y + 16 = 0$
22. (2, 1), (-2, -1)
23. $1/\sqrt{2}$
24. (4, 8)
25. Definition of a parabola: The locus of points equally distant from a fixed point and line.

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